



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

## Capillary surfaces arising in singular perturbation problems

**Citation for published version:**

Karakhanyan, A 2020, 'Capillary surfaces arising in singular perturbation problems', *Analysis & PDE*, vol. 13, no. 1, pp. 171-200. <https://doi.org/10.2140/apde.2020.13.171>

**Digital Object Identifier (DOI):**

[10.2140/apde.2020.13.171](https://doi.org/10.2140/apde.2020.13.171)

**Link:**

[Link to publication record in Edinburgh Research Explorer](#)

**Document Version:**

Peer reviewed version

**Published In:**

Analysis & PDE

**General rights**

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

**Take down policy**

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact [openaccess@ed.ac.uk](mailto:openaccess@ed.ac.uk) providing details, and we will remove access to the work immediately and investigate your claim.



# CAPILLARY SURFACES ARISING IN SINGULAR PERTURBATION PROBLEMS

ARAM L. KARAKHANYAN

ABSTRACT. In this paper we prove some Bernstein type theorems for a class of stationary points of the Alt-Caffarelli functional in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  arising as limits of the singular perturbation problem

$$\begin{cases} \Delta u_\varepsilon(x) = \beta_\varepsilon(u_\varepsilon) & \text{in } B_1, \\ |u_\varepsilon| \leq 1 & \text{in } B_1, \end{cases} \quad (0.1)$$

in the unit ball  $B_1$  as  $\varepsilon \rightarrow 0$ . Here  $\beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta(t/\varepsilon) \geq 0, \beta \in C_0^\infty[0, 1], \int_0^1 \beta(t)dt = M > 0$ , is an approximation of the Dirac measure and  $\varepsilon > 0$ . The limit functions  $u = \lim_{\varepsilon_j \rightarrow 0} u_{\varepsilon_j}$  of uniformly converging sequences  $\{u_{\varepsilon_j}\}$  solve a Bernoulli type free boundary problem in some weak sense. Our approach has two novelties: First we develop a hybrid method for stratification of the free boundary  $\partial\{u_0 > 0\}$  of blow-up solutions which combines some ideas and techniques of viscosity and variational theory. An important tool we use is a new monotonicity formula for the solutions  $u_\varepsilon$  based on a computation of J. Spruck. It implies that any blow-up  $u_0$  of  $u$  either vanishes identically or is homogeneous function of degree one, that is  $u_0 = rg(\sigma), \sigma \in \mathbb{S}^{N-1}$  in spherical coordinates  $(r, \theta)$ . In particular, this implies that in two dimensions the singular set is empty at the non-degenerate points, and in three dimensions the singular set of  $u_0$  is at most a singleton. Second, we show that the spherical part  $g$  is the support function (in Minkowski's sense) of some capillary surface contained in the sphere of radius  $\sqrt{2M}$ . In particular, we show that  $\nabla u_0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  is an almost conformal and minimal immersion and the singular Alt-Caffarelli example corresponds to a piece of catenoid which is a unique ring-type stationary minimal surface determined by the support function  $g$ .

## CONTENTS

1. Introduction	2
Related works	4
2. Notation	6
3. Monotonicity formula of Spruck	6
4. Proof of Theorem A	11
5. The structure of the free boundary of blow-ups in $\mathbb{R}^3$	12
5.1. Properties of $\partial\{u_0 > 0\}$	15
5.2. Weak solutions	18
5.3. Minimal perimeter	19
5.4. Full Non-degeneracy	19
5.5. Properties of $\Gamma_{\frac{1}{2}}$	20

---

2000 Mathematics Subject Classification. Primary 49Q05, 35R35, 35B25.

Keywords: Singular perturbation problem, free boundary regularity, capillary surfaces, global solutions.

5.6. Proof of Theorem B	21
6. Proof of Theorem C	21
6.1. Inverse Gauss map and the support function	21
6.2. Catenoid is a solution	22
6.3. Almost minimal immersions	23
6.4. Dual cones and center of mass	24
7. Appendix	27
References	29

## 1. INTRODUCTION

In this paper we study the singular perturbation problem

$$\begin{cases} \Delta u_\varepsilon(x) = \beta_\varepsilon(u_\varepsilon) & \text{in } B_1, \\ |u_\varepsilon| \leq 1 & \text{in } B_1, \end{cases} \quad (\mathcal{P}_\varepsilon)$$

where  $\varepsilon > 0$  is a small parameter and

$$\begin{cases} \beta_\varepsilon(t) = \frac{1}{\varepsilon} \beta\left(\frac{t}{\varepsilon}\right), \\ \beta(t) \geq 0, \quad \text{supp } \beta \subset [0, 1], \\ \int_0^1 \beta(t) dt = M > 0, \end{cases} \quad (1.1)$$

is an approximation of the Dirac measure,  $B_1 \subset \mathbb{R}^N$  is the unit ball centered at the origin. It is well known that  $(\mathcal{P}_\varepsilon)$  models propagation of equidiffusional premixed flames with high activation of energy [Caf95]. Heuristically, the limit  $u_0 = \lim_{\varepsilon_j \rightarrow 0} u_{\varepsilon_j}$  (for a suitable sequence  $\varepsilon_j \rightarrow 0$ ) solves a Bernoulli type free boundary problem with the following free boundary condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M.$$

If the functions  $\{u_\varepsilon\}$  are also minimizers of

$$J_\varepsilon[u_\varepsilon] = \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \mathcal{B}(u_\varepsilon/\varepsilon), \quad \mathcal{B}(t) = \int_0^t \beta(s) ds, \quad (1.2)$$

then the limits of  $\{u_\varepsilon\}$  inherit the generic features of minimizers (e.g. non-degeneracy, rectifiability of  $\partial\{u > 0\}$ , etc.). Consequently, the limits of uniformly converging sequences  $\{u_{\varepsilon_j}\}$  as  $\varepsilon_j \rightarrow 0$  are minimizers of the Alt-Caffarelli functional  $J[u] = \int_{B_1} |\nabla u|^2 + 2M\chi_{\{u>0\}}$ . It is known that the singular set of minimizers is empty in dimensions 2, 3 and 4, see [AC81], [CJK04], [JS15]. However, if  $u_\varepsilon$  is not a minimizer then the analysis of the limits  $u$  presents a more delicate problem. The main difficulty in carrying out such analysis is that the free boundary may contain degenerate points [Wei03].

This paper is devoted to the study of the blow-ups of the limits of the singular perturbation problem  $(\mathcal{P}_\varepsilon)$  and establishes a new and direct connection with **minimal surfaces**. In particular, we show that every blow-up of a limit function  $u = \lim_{\varepsilon_j \rightarrow 0} u_{\varepsilon_j}$  in  $\mathbb{R}^3$  (for an appropriate sequence  $\varepsilon_j$ ) defines an almost

conformal and minimal immersion which is perpendicular to the sphere of radius  $\sqrt{2M}$  where  $M = \int_0^1 \beta(t)dt$ . In other words, one obtains a capillary surface inside the sphere of radius  $\sqrt{2M}$ .

Our first result is

**Theorem A.** *Let  $u_{\varepsilon_j} \rightarrow u$  locally uniformly in  $B_1$  for some subsequence  $\varepsilon_j$ , then any blow-up of  $u$  at free boundary point  $x_0 \in \partial\{u > 0\}$  is either identically zero or homogeneous function of degree one. In particular, if  $N = 2$  and  $u$  is not degenerate at  $x_0 \in \partial\{u > 0\}$  then every blow-up of  $u$  at  $x_0$  must be one of the following functions (after some rotation of coordinates):*

- (1)  $\sqrt{2M}x_1^+$ , half plane solution provided that there is a measure theoretic normal at  $x_0$ ,
- (2) wedge  $\alpha|x_1|$ ,  $0 < \alpha \leq \sqrt{2M}$ ,
- (3) two plane solution  $\alpha x_1^+ - \beta x_1^-$ ,  $\alpha^2 - \beta^2 = 2M$ ,  $\alpha, \beta > 0$ .

In order to prove Theorem A we will introduce a monotone quantity based on a computation of Joel Spruck [Spr83]. From Theorem A it follows that in  $\mathbb{R}^2$  the blow-up limits at non-degenerate free boundary points can be explicitly computed. It is worthwhile to note that the minimizers of

$$J[u] = \int_{B_1} |\nabla u|^2 + 2M\chi_{\{u>0\}} \quad (1.3)$$

are non-degenerate, i.e. for each subdomain  $\Omega' \subset\subset B_1$  there is a constant  $c_0 > 0$  depending on  $\text{dist}(\partial B_1, \partial\Omega')$ ,  $N$ ,  $M$ , such that

$$\sup_{B_r(x_0)} u^+ \geq c_0 r, \quad \forall x_0 \in \partial\{u > 0\} \cap \Omega', \quad B_r(x_0) \subset B_1. \quad (1.4)$$

However, if  $u_\varepsilon$  is **any** solution of  $(\mathcal{P}_\varepsilon)$  then non-degeneracy may not be true. There is a sufficient condition [CLW97] Theorem 6.3 that implies (1.4).

Some well-known examples demonstrate rather strikingly that for the stationary case there are wedge-like global solutions for which the measure theoretic boundary of  $\{u > 0\}$  is empty. This is impossible for minimizers. In fact, the zero set of a minimizer has uniformly positive Lebesgue density. In this respect Theorem A only states that if  $u$  is non-degenerate at  $x_0$  then the blow-up is a nontrivial cone.

The existence of wedge solutions (see Remark 5.1 [CLW97]) suggests that some further assumptions are needed to formulate the free boundary condition. For instance, one may assume that the upper Lebesgue density at  $x \in \partial\{u > 0\}$  satisfies  $\Theta^*(x, \{u > 0\}) < 1$ , i.e. the upper density measure is not covering the full ball. We emphasize that for some solutions the topological and measure theoretic boundaries may not coincide. Our next result addresses the degeneracy and wedge-formation in  $\mathbb{R}^3$  of blow-ups at free boundary points.

**Theorem B.** *Suppose  $N = 3$ . Let  $u \geq 0$  be a limit of some uniformly converging sequence  $\{u_{\varepsilon_j}\}$  solving  $(\mathcal{P}_\varepsilon)$  such that  $u$  is non-degenerate at  $y_0 \in \partial\{u > 0\}$ . Let  $u_0$  be a blow-up of  $u$  at  $y_0$ . If  $\mathfrak{C}$  is a component of  $\partial\{u_0 > 0\}$  such that the measure theoretic boundary of  $\{u_0 > 0\}$  in  $\mathfrak{C}$  is non-empty then*

- (1) all points of  $\mathfrak{C}$  are non-degenerate,
- (2)  $\mathfrak{C}$  is a subset of the measure theoretic boundary of  $\{u_0 > 0\}$ ,
- (3)  $\mathfrak{C} \setminus \{0\}$  is smooth.

In particular in  $\mathbb{R}^3$  the singular set of  $\partial\{u_0 > 0\}$  is at most a singleton.

Theorem B implies that the reduced boundary propagates instantaneously in the components of  $\partial\{u_0 > 0\}$ . Our last result sheds some new light on the characterization of the blow-ups as minimal surfaces inside spheres with contact angle  $\pi/2$ .

**Theorem C.** *Let  $u_0$  be as in Theorem B and  $u_0 = rg(\sigma), \sigma \in \mathbb{S}^2$  in spherical coordinates. Then the parametrization  $\mathcal{X}(\sigma) = \sigma g(\sigma) + \nabla_{\mathbb{S}^2} g(\sigma)$  defines an almost conformal and minimal immersion. If  $\{g > 0\}$  is homeomorphic to a disk then  $u_0$  is a half-plane solution  $\sqrt{2M}x_1^+$ . If  $\{g > 0\}$  is homeomorphic to a ring then the only singular cone is the Alt-Caffarelli catenoid.*

Observe that  $\Delta u_0 = 0$  implies that the spherical part  $g$  satisfies the following equation on the sphere

$$\Delta_{\mathbb{S}^{N-1}} g + (N-1)g = 0,$$

where  $\Delta_{\mathbb{S}^{N-1}}$  is the Laplace-Beltrami operator. If we regard  $g$  as the support function of some embedded hypersurface  $\mathcal{M}$  then the matrix  $[\nabla_{ij}g + \delta_{ij}g]^{-1}$  gives the Weingarten mapping and its eigenvalues are the principal curvatures  $k_1, \dots, k_{N-1}$  of  $\mathcal{M}$ . If  $N = 3$  then we have that

$$0 = \Delta_{\mathbb{S}^2} g + 2g = \text{trace}[\nabla_{ij}g + \delta_{ij}g] = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_1 + k_2}{k_1 k_2}$$

implying that the mean curvature is zero at the points where the Gauss curvature  $k_1 k_2$  does not vanish. This is how the minimal surfaces enter into the game. One of the main obstacles is to show that the surface parametrized by  $\mathcal{X}(\sigma) = \nabla u_0(\sigma)$  is embedded. Then the classification for the disk-type domains  $\{g > 0\}$  follows from a result of Nitsche [Nit85]. To prove the last statement of Theorem C we will use the moving plane method. It is worthwhile to point out that the results of this paper can be extended to other classes of stationary points. For instance, the weak solutions introduced in [AC81] can be analyzed in similar way provided that the zero set has uniformly positive Lebesgue density at free boundary points in order to guarantee that the class of weak solution is closed with respect to blow-ups, see example 5.8 in [AC81].

**Related works.** In [HHP11] F. Hélein, L. Hauswirth and F. Pacard have considered the following overdetermined problem

$$\begin{cases} \Delta u(x) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0, |\nabla u| = 1 & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

where  $\Omega$  is a smooth domain and the boundary conditions are satisfied in the classical sense. A domain  $\Omega$  admitting a solution  $u$  to (1.5) is called exceptional. Note that every nonnegative smooth solution of the limiting singular perturbation problem solves (1.5) with  $M = \frac{1}{2}$ . In [HHP11] the authors have constructed a number of examples of exceptional domains and proposed to classify them. In particular, they proved that if  $\Omega \subset \mathbb{R}^2$  is conformal to half-plane such that  $u$  is strictly monotone in one fixed direction then  $\Omega$  is a half-space, [HHP11, Proposition 6.1]. However the general problem remained open.

Later M. Traizet showed that the smoothness assumption can be relaxed, namely if  $\Omega \subset \mathbb{R}^2$  has  $C^0$  boundary and the boundary conditions are still satisfied in the classical sense then  $\Omega$  is real analytic

[Tra14, Proposition 1]. Under various topological conditions on the two dimensional domain  $\Omega \subset \mathbb{R}^2$  (such as finite connectivity and periodicity) M. Traizet classified the possible exceptional domains. One of his remarkable results is that from  $\Omega$  one can construct a *complete* minimal surface using the Weierstrass representation formula [Tra14, Theorem 9]. Another classification result in  $\mathbb{R}^2$ , under stronger topological hypotheses than in [Tra14], is given by D. Khavinson, E. Lundberg and R. Teodorescu [KLT13]. Moreover, their results in simply connected case are stronger because unlike M. Traizet they do not assume the finite connectivity (i.e.  $\partial\Omega$  has finite number of components). As opposed to these results (1) we do not assume any regularity of the free boundary (which plays the role of  $\partial\Omega$  in (1.5)), (2) the Neumann condition is not satisfied in the classical sense, (3) the minimal surface we construct in Theorem C is *not complete* and it is a capillary surface inside sphere, and (4) our techniques do not impose any restriction on the dimension. Note that, in [HHP11] the authors suggested to study more general classes of exceptional domains: if  $(M, g)$  is an  $m$ -dimensional Riemannian manifold admitting a harmonic function with zero Dirichlet and constant Neumann boundary data then  $M$  is called exceptional and  $u$  a roof function. In this context Theorem C provides a way of constructing roof function on the sphere from the blow-ups of stationary points of the Alt-Caffarelli functional.

One may consider higher order critical points as well, such as mountain passes (which are, in fact, minimizers over some subspace of admissible functions) for which one has non-degeneracy and nontrivial Lebesgue density properties [DJKP16, Propositions 1.7-5.1]. Observe that neither of these properties is available for our solutions as Theorem 6.3 and Remark 5.1 in [CLW97] indicate, and in the present work we do not impose any additional assumptions on our stationary points of this kind.

It seems that the only result in high dimensions that appears in [HHP11], [KLT13] and [Tra14] states that if the complement of  $\Omega$  is connected and has  $C^{2,\alpha}$  boundary, then  $\Omega$  is the exterior of a ball [KLT13, Theorem 7.1]. The restriction  $\Omega \subset \mathbb{R}^2$  is because the authors have mainly used the techniques from complex analysis. Our approach does not have this restriction since our main tool is the representation of the solution in terms of the Minkowski support function. We remark that using our method in high dimensions we can construct a surface  $\mathcal{M}$  inside the sphere of radius  $\sqrt{2M}$  such that the sum of its principal radii of curvature is zero, and  $\mathcal{M}$  is transversal to the sphere.

Finally, we point out that our approach may lead to a new characterization of global minimizers in  $\mathbb{R}^3$  [CJK04]. Indeed, Theorem 6 from [RV95] implies that the capillary surface  $\mathcal{M}$  in Theorem C associated with the blow-up limit must be totally geodesic (i.e. the second fundamental form is identically zero). Consequently, the blow-up must be the half-plane solution.

The paper is organized as follows: In Section 2 we set up some basic notation which will be used throughout the paper. Section 3 is devoted to the study of a new monotone quantity  $s(x_0, u, r)$ . This interesting quantity is derived from a computation of J. Spruck [Spr83]. Among other things, properties of  $s$  imply that every blow-up of  $u$  is either homogeneous function of degree one or identically zero. Section 4 contains the proof of Theorem A. In Section 5 we develop a new method of stratification of the free boundary points and prove Theorem B. Section 6 contains the proof of Theorem C. For the convenience of the reader in Appendix we repeat the relevant material from [CLW97] without proofs.

## 2. NOTATION

Throughout the paper  $N$  will denote the spatial dimension.  $B_r(x_0) = \{x \in \mathbb{R}^N, s.t. |x - x_0| < r\}$  denotes the open ball of radius  $r > 0$  centered at  $x_0 \in \mathbb{R}^N$ . The  $s$ -dimensional Hausdorff measure is denoted by  $\mathcal{H}^s$ , the unit sphere by  $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ , and the characteristic function of the set  $D$  by  $\chi_D$ . We also

$$M = \int_0^1 \beta(t) dt.$$

Sometimes we will denote  $x = (x_1, x')$  where  $x' \in \mathbb{R}^{N-1}$ . For given function  $v$ , we will denote  $v^+ = \max(0, v)$  and  $v^- = \max(0, -v)$ . Finally, we say that  $v \in C_{loc}^{0,1}(\mathcal{D})$  if for every  $\mathcal{D}' \Subset \mathcal{D}$ , there is a constant  $L(\mathcal{D}')$  such that

$$|v(x) - v(y)| \leq L(\mathcal{D}')|x - y|, \quad \forall x, y \in \mathcal{D}.$$

If  $v \in C_{loc}^{0,1}(\mathcal{D})$  then we say that  $v$  is locally Lipschitz continuous in  $\mathcal{D}$ . For  $x = (x_1, \dots, x_N)$  and fixed  $x_0 \in \mathbb{R}^N$   $(x - x_0)_1^+$  denotes the positive part of the first coordinate of  $x - x_0$ . If  $u(x_0) = 0$  then  $(u(x))_r = u(x_0 + rx)/r, r > 0$  denotes the scaled function at  $x_0$ . For given  $r_j \rightarrow 0$  the sequence  $(u(x))_{r_j}$  is called a blow-up sequence and its limit  $u_0$  a blow-up of  $u$  at  $x_0$ .

## 3. MONOTONICITY FORMULA OF SPRUCK

It is convenient to work with a weaker definition of non-degeneracy which only assures that the blow-up does not vanish identically.

**Definition 3.1.** *We say that  $u$  is degenerate at  $x_0 \in \partial\{u > 0\}$  if  $\liminf_{r \rightarrow 0} \frac{1}{r} \int_{B_r(x_0)} u^+ = 0$ .*

Observe that  $u^+(x) = o(|x - x_0|)$  near the degenerate point  $x_0$  because  $u^+$  is subharmonic.

It is known that the solutions of  $(\mathcal{P}_\varepsilon)$  are locally Lipschitz continuous, see Appendix Proposition 7.1. Consequently, there is a subsequence  $\varepsilon_j \rightarrow 0$  such that  $u_{\varepsilon_j} \rightarrow u$  locally uniformly. Furthermore,  $u$  is a stationary point of the Alt-Caffarelli problem in some weak sense and the blow-up of  $u$  can be approximated by some scaled family of solutions to  $(\mathcal{P}_\varepsilon)$ , see Appendix Propositions 7.5 and 7.6.

**Proposition 3.1.** *Let  $u$  be a limit of some sequence  $u_{\varepsilon_j}$  as in Proposition 7.2. Then any blow-up of  $u$  at a non-degenerate point is a homogeneous function of degree one.*

**Proof.** To fix the ideas we assume that  $0 \in \partial\{u > 0\}$  is a non-degenerate point. We begin with writing the Laplacian in polar coordinates

$$\Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} u \quad (3.1)$$

and then introducing the auxiliary function

$$v(t, \sigma) = \frac{u(r, \sigma)}{r}, \quad r = e^{-t}. \quad (3.2)$$

A straightforward computation yields

$$\begin{aligned} v_t &= -u_r + v, \\ v_\sigma &= \frac{u_\sigma}{r}, \\ v_{tt} &= u_{rr}r + v_t, \\ \Delta_{\mathbb{S}^{N-1}} v &= \frac{1}{r} \Delta_{\mathbb{S}^{N-1}} u, \end{aligned}$$

where, with some abuse of notation,  $v_\sigma$  denotes the gradient of  $v$  computed on the sphere. Rewriting the equation  $\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$  in  $t$  and  $\sigma$  derivatives we obtain

$$\frac{1}{r} [(N-1)(v - \partial_t v_\varepsilon) + \partial_t^2 v_\varepsilon - \partial_t v_\varepsilon + \Delta_{\mathbb{S}^{N-1}} v_\varepsilon] = \frac{1}{\varepsilon} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right).$$

Next, we multiply both sides of the last equation by  $\partial_t v_\varepsilon$  to get

$$\partial_t v_\varepsilon [(N-1)(v - \partial_t v_\varepsilon) + \partial_t^2 v_\varepsilon - \partial_t v_\varepsilon + \Delta_{\mathbb{S}^{N-1}} v_\varepsilon] = \partial_t v_\varepsilon \frac{r}{\varepsilon} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right). \quad (3.3)$$

The right hand side of (3.3) can be further transformed as follows

$$\begin{aligned} \frac{r}{\varepsilon} \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \partial_t v_\varepsilon &= \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \left[ \frac{e^{-t}}{\varepsilon} \partial_t v_\varepsilon - \frac{e^{-t}}{\varepsilon} v_\varepsilon \right] + \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \\ &= \partial_t \int_0^{\frac{e^{-t}}{\varepsilon} v_\varepsilon} \beta(s) ds + \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \\ &= \partial_t \mathcal{B} \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) + \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \\ &\equiv I_1. \end{aligned}$$

It is important to note that by our assumption (1.1) the last term is nonnegative, in other words

$$\beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \geq 0. \quad (3.4)$$

Moreover, we have

$$\begin{aligned} I_2 &\equiv [(N-1)v_\varepsilon - N\partial_t v_\varepsilon + \partial_t^2 v_\varepsilon + \Delta_{\mathbb{S}^{N-1}} v_\varepsilon] \partial_t v_\varepsilon \\ &= (N-1)\partial_t \left( \frac{v_\varepsilon^2}{2} \right) - N(\partial_t v_\varepsilon)^2 + \partial_t \left( \frac{(\partial_t v_\varepsilon)^2}{2} \right) + \partial_t v_\varepsilon \Delta_{\mathbb{S}^{N-1}} v_\varepsilon. \end{aligned}$$

Next we integrate the identity

$$I_2 = rI_1$$

over  $\mathbb{S}^{N-1}$  and then over  $[T_0, T]$  in order to get

$$\begin{aligned} (N-1) \int_{\mathbb{S}^{N-1}} \frac{v_\varepsilon^2}{2} \Big|_{T_0}^T - N \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 + \int_{\mathbb{S}^{N-1}} \frac{(\partial_t v_\varepsilon)^2}{2} \Big|_{T_0}^T + \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \partial_t v_\varepsilon \Delta_{\mathbb{S}^{N-1}} v_\varepsilon \\ = \int_{\mathbb{S}^{N-1}} \mathcal{B} \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \Big|_{T_0}^T + \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right) \frac{r}{\varepsilon} v_\varepsilon. \end{aligned}$$



Note that

$$\int_{T_0}^T \int_{\mathbb{S}^{N-1}} \partial_t v_\varepsilon \Delta_{\mathbb{S}^{N-1}} v_\varepsilon = - \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\nabla_\sigma v_\varepsilon|^2 \Big|_{T_0}^T. \quad (3.5)$$

Rearranging the terms and utilizing (3.4) we get the identity

$$\begin{aligned} N \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 + \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right) \frac{r}{\varepsilon} v_\varepsilon &= (N-1) \int_{\mathbb{S}^{N-1}} \frac{v_\varepsilon^2}{2} \Big|_{T_0}^T + \int_{\mathbb{S}^{N-1}} \frac{(\partial_t v_\varepsilon)^2}{2} \Big|_{T_0}^T \\ &\quad - \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\nabla_\sigma v_\varepsilon|^2 \Big|_{T_0}^T - \int_{\mathbb{S}^{N-1}} \mathcal{B} \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \Big|_{T_0}^T. \end{aligned} \quad (3.6)$$

From here it follows that

$$\int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 \leq C \quad (3.7)$$

where  $C$  depends on  $\|\nabla u_\varepsilon\|_\infty, M, N$  but not on  $\varepsilon, T_0$  or  $T$ .

Letting  $\varepsilon \rightarrow 0$  we conclude

$$\int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v)^2 \leq C \quad (3.8)$$

where  $v(t, \sigma) = \frac{u(r, \sigma)}{r}$ . But  $\partial_t v = -u_r + \frac{u}{r}$  implying that

$$\int_{T_0}^\infty \int_{\mathbb{S}^{N-1}} \left( u_r - \frac{u}{r} \right)^2 dt d\sigma \leq C. \quad (3.9)$$

The proof of Theorem A follows if we note that  $-u_r + \frac{u}{r} = 0$  is the Euler equation for the homogeneous functions of degree one.  $\square$

In the proof of Proposition 3.1 we used Spruck's original computation [Spr83]. The identity (3.6) can be interpreted as a local energy balance for  $u_\varepsilon$ . Moreover, using (3.6) we can construct a monotone quantity which has some remarkable properties.

**Corollary 3.2.** *Suppose  $0 \in \partial\{u > 0\}$  and let  $(r, \sigma), \sigma \in \mathbb{S}^{N-1}$  be the spherical coordinates. Introduce*

$$S_\varepsilon(r) = \int_{\mathbb{S}^{N-1}} \left\{ 2\mathcal{B}(u_\varepsilon(r, \sigma)/\varepsilon) + \frac{1}{r^2} |\nabla_\sigma u_\varepsilon|^2 - (N-1) \frac{u_\varepsilon^2(r, \sigma)}{r^2} - \left( \partial_r u_\varepsilon(r, \sigma) - \frac{u_\varepsilon(r, \sigma)}{r} \right)^2 \right\} d\sigma. \quad (3.10)$$

- Then  $S_\varepsilon(r)$  is nondecreasing in  $r$ .
- Moreover, if  $u_{\varepsilon_j} \rightarrow u$  for some subsequence  $\varepsilon_j \rightarrow 0$ , then  $S_{\varepsilon_j}(r) \rightarrow S(r)$  for a.e.  $r$  where

$$S(r) = \int_{\mathbb{S}^{N-1}} \left\{ 2M\chi_{\{u>0\}} + \frac{1}{r^2} |\nabla_\sigma u|^2 - (N-1) \frac{u^2(r, \sigma)}{r^2} - \left( \partial_r u(r, \sigma) - \frac{u(r, \sigma)}{r} \right)^2 \right\} d\sigma. \quad (3.11)$$

In particular,  $S(r)$  is nondecreasing function of  $r$ .

- $S(r)$  is constant if and only if  $u$  is a homogenous function of degree one.

**Proof.** By setting  $r_1 = e^{-T}$ ,  $r_2 = e^{-T_0}$  and noting that  $r_1 < r_2$  if  $T > T_0$  we obtain from (3.6)

$$S_\varepsilon(r_2) - S_\varepsilon(r_1) = 2N \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 + 2 \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \beta\left(\frac{r}{\varepsilon} v_\varepsilon\right) \frac{r}{\varepsilon} v_\varepsilon \geq 0$$

where we applied (3.4) and hence the first claim follows. The second part follows from Propositions 7.1 and 7.2. Indeed, integrating  $S_\varepsilon(r) \leq S_\varepsilon(r+t)$ ,  $t \geq 0$  over  $[r_1 - \delta, r_1 + \delta]$  we infer

$$\frac{1}{2\delta} \int_{r_1 - \delta}^{r_1 + \delta} S_\varepsilon(r) dr \leq \frac{1}{2\delta} \int_{r_1 - \delta}^{r_1 + \delta} S_\varepsilon(r+t) dr.$$

Then first letting  $\varepsilon \rightarrow 0$  and utilizing Proposition 7.2 together with (7.1) and then sending  $\delta \rightarrow 0$  we infer that  $S(r)$  is nondecreasing for a.e.  $r$ . Finally the last part follows as in the proof of Proposition 3.1.  $\square$

As one can see we did not use the Pohozaev identity as opposed to the monotonicity formula in [Wei03]. Spruck's monotonicity formula enjoys a remarkable property.

**Lemma 3.3.** *Let  $u$  be as in Proposition 3.1. Set  $S(x_0, r, u)$  for  $S(r)$  defined by the sphere centered at  $x_0 \in \partial\{u > 0\}$ . Suppose  $x_k \in \partial\{u > 0\}$  such that  $x_k \rightarrow x_0$  then*

$$\limsup_{x_k \rightarrow x_0} S(x_k, 0, u) \leq S(x_0, 0, u).$$

**Proof.** For given  $\delta > 0$  there is  $\rho_0 > 0$  such that  $S(x_0, \rho, u) \leq S(x_0, 0, u) + \delta$  whenever  $\rho < \rho_0$ . Fix such  $\rho$  and choose  $k$  so large that  $S(x_k, \rho, u) < \delta + S(x_0, \rho, u)$ . From the monotonicity of  $S(x_k, \rho, u)$  it follows that

$$\begin{aligned} S(x_k, 0, u) &\leq S(x_k, \rho, u) \leq \delta + S(x_0, \rho, u) \\ &\leq 2\delta + S(x_0, 0, u). \end{aligned}$$

First letting  $x_k \rightarrow x_0$  and then  $\delta \rightarrow 0$  the result follows.  $\square$

**Lemma 3.4.** *Let  $S$  be the monotone quantity in (3.11). Then the following holds:*

(i)  $s(x_0, R, u) = \frac{1}{R^N} \int_0^R r^{N-1} S(x_0, r, u) dr$  is monotone non-decreasing and

$$\frac{d}{dR} s(x_0, u, R) = \frac{1}{R^{N+1}} \int_0^R r^N S'(x_0, u, r) dr \geq 0.$$

(ii) If the solution  $u \geq 0$  is degenerate at  $x_0 \in \partial\{u > 0\}$  then the set  $\{u > 0\}$  has well defined Lebesgue density  $\Theta(x_0, \{u > 0\})$  equal to  $\frac{1}{2M|B_1|} s(x_0, 0, u) = \frac{1}{2M|B_1|} \lim_{R \rightarrow 0} s(x_0, R, u)$ .

(iii) Suppose  $x_k \in \partial\{u > 0\}$  such that  $x_k \rightarrow x_0$  then

$$\limsup_{x_k \rightarrow x_0} s(x_k, 0, u) \leq s(x_0, 0, u).$$

**Proof.** It is easy to compute

$$\begin{aligned}
s'(x_0, R, u) &= -\frac{N}{R^{N+1}} \int_0^R r^{N-1} S(x_0, r, u) dr + \frac{S(x_0, R, u)}{R} \\
&= -\frac{S(x_0, R, u)}{R} + \frac{1}{R^{N+1}} \int_0^R r^N S'(x_0, u, r) dr + \frac{S(x_0, R, u)}{R} \\
&= \frac{1}{R^{N+1}} \int_0^R r^N S'(x_0, r, u) dr.
\end{aligned}$$

To prove the second claim notice that at degenerate point  $x_0$  we have  $u(x) = o(|x - x_0|)$  by virtue of the subharmonicity of  $u$ . Consequently  $\int_{B_R(x_0)} |\nabla u|^2 = o(1)$  as  $r \rightarrow 0$  by virtue of the Caccioppoli inequality. Therefore the only surviving term in  $S$  comes from  $2M\chi_{\{u>0\}}$ . The proof of the last claim is analogous to that of Lemma 3.3.  $\square$

**Lemma 3.5.** *Let  $0 \in \partial\{u > 0\}$  and assume that  $u_0 = rg(\sigma), \sigma \in \mathbb{S}^{N-1}$  is a blow-up limit of  $u$  at 0 which is homogeneous function of degree one. Then*

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}} |\nabla_\sigma g|^2 - (N-1) \int_{\mathbb{S}^{N-1}} g^2 &= 0 & \text{if } \partial\{u > 0\} \text{ is flat at 0,} \\
&\leq 0 & \text{otherwise.}
\end{aligned}$$

**Proof.** Let  $(r, \sigma)$  be the spherical coordinates then the Laplacian takes the form  $\Delta u_\varepsilon = \partial_{rr}^2 u_\varepsilon + \frac{N-1}{r} \partial_r u_\varepsilon + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} u_\varepsilon$ . Multiply both sides of  $\Delta u_\varepsilon$  by  $r^{N-1} u_\varepsilon$  and integrate over  $[0, R] \times \mathbb{S}^{N-1}$  to get

$$\begin{aligned}
I_1(u_{\varepsilon_j}) &:= \int_0^R \int_{\mathbb{S}^{N-1}} u_\varepsilon \partial_{rr}^2 u_\varepsilon r^{N-1} d\sigma dr \\
&= R^{N-1} \int_{\mathbb{S}^{N-1}} u_\varepsilon \partial_r u_\varepsilon - \int_0^R \int_{\mathbb{S}^{N-1}} [(\partial_r^2 u_\varepsilon)^2 r^{N-1} + (N-1) \partial_r u_\varepsilon u_\varepsilon r^{N-1}] d\sigma dr, \\
I_2(u_{\varepsilon_j}) &:= \int_0^R \int_{\mathbb{S}^{N-1}} u_\varepsilon \partial_r^2 u_\varepsilon r^{N-2} d\sigma dr = R^{N-2} \int_{\mathbb{S}^{N-1}} \frac{(u_\varepsilon)^2}{2} - (N-2) \int_0^R \int_{\mathbb{S}^{N-1}} \left[ \frac{(u_\varepsilon)^2}{2} r^{N-3} \right] d\sigma dr, \\
I_3(u_{\varepsilon_j}) &:= \int_0^R \int_{\mathbb{S}^{N-1}} \Delta_{\mathbb{S}^{N-1}} u_\varepsilon u_\varepsilon r^{N-3} d\sigma dr = - \int_0^R \int_{\mathbb{S}^{N-1}} |\nabla_\sigma u_\varepsilon|^2 r^{N-3} d\sigma dr.
\end{aligned}$$

Choosing a converging sequence  $u_{\varepsilon_j}$  and letting  $\varepsilon_j \rightarrow 0$  we get by virtue of Proposition 7.2

$$\lim_{\varepsilon_j \rightarrow 0} \int_{B_R} \beta_{\varepsilon_j} u_{\varepsilon_j} = \lim_{\varepsilon_j \rightarrow 0} [I_1(u_{\varepsilon_j}) + (N-1)I_2(u_{\varepsilon_j}) + I_3(u_{\varepsilon_j})] \rightarrow I_1(u) + (N-1)I_2(u) + I_3(u).$$

Suppose that  $u_{R_k}$  is a blow-up sequence at the origin and  $u_{R_k} \rightarrow u_0 = rg(\sigma)$  then

$$I_1(u_0) = R^N \int_{\mathbb{S}^{N-1}} g^2 - \frac{R^N}{N} \int_{\mathbb{S}^{N-1}} g^2 - \frac{N-1}{N} R^N \int_{\mathbb{S}^{N-1}} g^2 = 0$$

and

$$I_2(u_0) = R^N \int_{\mathbb{S}^{N-1}} \frac{g^2}{2} - \frac{N-2}{N} R^N \int_{\mathbb{S}^{N-1}} \frac{g^2}{2} = \frac{R^N}{N} \int_{\mathbb{S}^{N-1}} g^2.$$

By Proposition 7.5 and (7.2) there is a sequence  $\delta_j \rightarrow 0$  such that  $u_{\delta_j} \rightarrow u_0$  and  $\lim_{\delta_j \rightarrow 0} \int_{B_1} \beta_{\delta_j} u_{\delta_j} \leq \|\beta\|_\infty |x \in B_1 : 0 < u_{\delta_j} < \delta_j| \rightarrow 0$  provided that  $u$  is flat at 0. Hence we have

$$\lim_{\delta_j \rightarrow 0} \int_{B_1} \beta_{\delta_j} u_{\delta_j} = \frac{R^N}{N} \left[ (N-1) \int_{\mathbb{S}^{N-1}} g^2 - \int_{\mathbb{S}^{N-1}} |\nabla_\sigma g|^2 \right] \begin{array}{ll} = 0 & \text{if } \partial\{u > 0\} \text{ is flat at 0,} \\ \geq 0 & \text{otherwise.} \end{array}$$

□

#### 4. PROOF OF THEOREM A

The first part of the theorem follows from Proposition 3.1. Since  $u$  is not degenerate at the origin then by Propositions 7.2 and 7.5  $u_{\rho_k}(x) \rightarrow u_0(x)$  locally uniformly and by Proposition 3.1  $u_0$  is homogeneous of degree one. Write  $\Delta$  in polar coordinates  $(r, \theta)$ , to obtain that

$$\Delta w = \frac{1}{r} \frac{\partial}{\partial r} (r w_r) + \frac{1}{r^2} \frac{\partial}{\partial \theta} (w_\theta).$$

In particular, writing  $u_0 = r g(\theta)$ , this yields a second order ODE for  $g$

$$g + \ddot{g} = 0. \quad (4.1)$$

Suppose  $g(0) = g(\theta_0) = 0, \theta_0 \in [0, 2\pi)$ , then (4.1) implies that  $g(\theta) = A \sin \theta$  for some constant  $A$ , and consequently forcing  $\theta_0 = \pi$ . Hence, since  $N = 2$ , we obtain that  $u_0$  must be linear, in other words the free boundary  $\partial\{u_0 > 0\}$  is everywhere flat. This in turn implies that in two dimensions the singular set of the free boundary  $\partial\{u_0 > 0\}$  is empty. Consequently,  $u_0$  is linear in  $\{u_0 > 0\}$  and  $\{u_0 < 0\}$ . From here the parts (2) and (3) of Theorem A follow from Propositions 5.3 and 5.1 of [CLW97].

So it remains to check (1). For the elliptic problem the only difference is that the limit function  $M(x) = \lim_{\delta_j \rightarrow 0} \mathcal{B}_{\delta_j}(u_{\delta_j})$  cannot have nontrivial concentration on the free boundary coming from  $\{x_1 < 0\}$  as opposed to the parabolic case studied in [CLW97]. Observe that  $\nabla \mathcal{B}(u_{\delta_j}/\delta_j) = \nabla u_{\delta_j} \beta_{\delta_j}(u_{\delta_j}) = 0$  in  $B_1 \setminus \{0 < u_{\delta_j} < \delta_j\}$ . By Proposition 7.5 and (7.2) there is sequence  $0 < \lambda_j \rightarrow 0$  such that  $(u_{\varepsilon_j})_{\lambda_j} \rightarrow u_0, \varepsilon_j/\lambda_j \rightarrow 0$  and  $M(x) = M \chi_{\{x_1 > 0\}} + M_0 \chi_{\{x_1 < 0\}}$ . It follows from (7.3) that

$$\int_{\{x_1 > 0\}} M \partial_1 \phi + \int_{\{x_1 < 0\}} M_0 \partial_1 \phi = \int_{\{x_1 > 0\}} \frac{\alpha^2}{2} \partial_1 \phi, \quad \forall \phi \in C_0^\infty(B_1). \quad (4.2)$$

After integration by parts we obtain  $M_0 \int_{-1}^1 \phi(0, x_2) dx_2 = \left(M - \frac{\alpha^2}{2}\right) \int_{-1}^1 \phi(0, x_2) dx_2$ . This yields

$$M_0 = M - \frac{\alpha^2}{2}.$$

Next we claim that  $M_0 = 0$ . Suppose  $M_0 > 0$  then the set  $I_0 := \{t \in \mathbb{R} : \mathcal{B}(t) = M_0\} \neq \emptyset$  and there is  $a \in (0, 1)$  such that  $I_0 \subset [a, 1]$ . Since  $\mathcal{B}(t)$  is continuous and non-decreasing then it follows that there is  $0 < a_0 < a$  such that  $u_{\delta_j}(x)/\delta_j \in [a_0, 1]$  provided that  $j$  is sufficiently large. Let  $\mathcal{C} = \{x : u_{\delta_j}(x)/\delta_j \in [a_0, 1]\} \cap \{x_1 < 0\} \cap B_1$ . Then,  $\mathcal{C} \subset \{x \in B_1 : a_0 \delta_j \leq u_{\delta_j}(x) \leq \delta_j\} \subset \{0 < u_{\delta_j} < 2\delta_j\} \cap B_1$ . But  $|\{0 < u_{\delta_j} < 2\delta_j\} \cap B_1| \rightarrow 0$  which implies that  $M_0$  cannot be positive. □

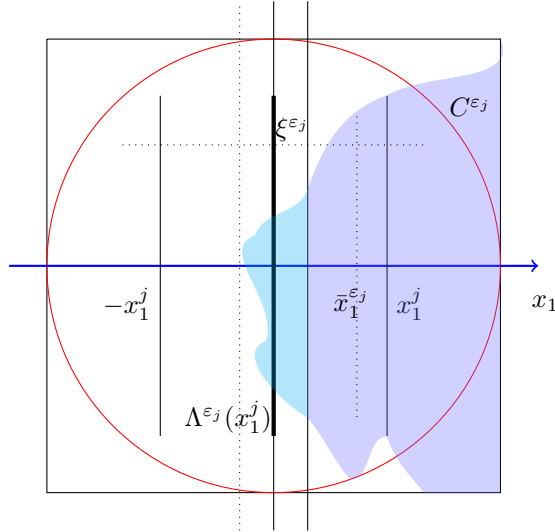


FIGURE 1. The construction of the point  $(\bar{x}_1^{\varepsilon_j}, \xi^{\varepsilon_j})$ . The purple region is  $C^{\varepsilon_j}$ .

## 5. THE STRUCTURE OF THE FREE BOUNDARY OF BLOW-UPS IN $\mathbb{R}^3$

In this section we assume that  $u \geq 0$  is a limit of  $u_{\varepsilon_j}$  solving  $(\mathcal{P}_\varepsilon)$  for some sequence  $\varepsilon_j \rightarrow 0$ ,  $u$  is non-degenerate at some  $y_0 \in \partial\{u > 0\}$  and  $u_0$  is a blow-up of  $u$  at  $y_0$ . Note that by Corollary 3.2  $u_0$  is homogenous function of degree one. If  $u_0$  is not a minimizer then it is natural to expect that the solutions of  $(\mathcal{P}_\varepsilon)$  develop singularities in  $\mathbb{R}^N$ ,  $N \geq 3$ .

We first prove a non-degeneracy result.

**Lemma 5.1.** *Let  $x_0 \in \partial\{u_0 > 0\}$  be a free boundary point such that there is a ball  $B \subset \{u_0 = 0\}$  touching  $\partial\{u_0 > 0\}$  at  $x_0$  and  $\Theta(x_0, \{u_0 > 0\}) \geq \frac{1}{2}$ . Then  $u_0$  is non-degenerate at  $x_0$  and*

$$u_0(x) = \sqrt{2M}(x - x_0)^+ + o(x - x_0).$$

**Proof.** Let  $(u_0)_r = \frac{u_0(x_0 + rx)}{r}$ . There is  $r_0$  such that

$$(u_0)_r = 0 \quad \text{in } \{x_1 < -\delta\} \cap Q_1, \quad \forall r \leq r_0 \quad (5.1)$$

for some small  $\delta > 0$ , where  $Q_1 = (-1, 1)^3$  is the unit cube. Moreover, there is  $\hat{r}_0 > 0$  such that

$$\frac{|\{(u_0)_r > 0\} \cap \{x_1 > 0\} \cap B_1|}{|B_1|} > \frac{1}{2} - \delta, \quad \forall r \leq \hat{r}_0. \quad (5.2)$$

Fix  $r$  with these two properties (5.1) and (5.2). There exists  $\gamma > 0$  such that

$$\frac{|\{(u_0)_r > \gamma\} \cap \{x_1 > 0\} \cap B_1|}{|B_1|} > \frac{1}{2} - 2\delta \quad (5.3)$$

Denote  $v^{\varepsilon_j} = (u_{\varepsilon_j})_r$  where  $u_{\varepsilon_j} \rightarrow u_0$  (see Proposition 7.5) and  $A^{\varepsilon_j} = \{v^{\varepsilon_j} > \frac{\gamma}{2}\} \cap \{x_1 > 0\} \cap B_1$ . Since  $v^{\varepsilon_j} \rightarrow (u_0)_r$  uniformly (see Proposition 7.2) it follows that there is  $j_0(r)$  such that for  $j \geq j_0(r)$  we have

$$|A^{\varepsilon_j}| > |B_1|(\frac{1}{2} - 2\delta). \quad (5.4)$$

Let  $B^{\varepsilon_j} = \{x_1 \in (-1, -\delta)\} \cap Q_1$  and  $-B^{\varepsilon_j} = \{x_1 \in (\delta, 1)\} \cap Q_1$ . Let  $C^{\varepsilon_j} = A^{\varepsilon_j} \cap (-B^{\varepsilon_j})$  then we have that

$$|C^{\varepsilon_j}| \geq |B_1|(\frac{1}{2} - 2\delta) > 0.$$

Denote  $\Lambda^{\varepsilon_j}(x_1) = \{x' : (x_1, x') \in C^{\varepsilon_j}\}$  and  $f^{\varepsilon_j}(x_1) = |\Lambda^{\varepsilon_j}(x_1)|$ . We claim that  $|\{x_1 : f^{\varepsilon_j}(x_1) > |B_1|(\frac{1}{2} - 3\delta)\}| > 0$ . Indeed, if the claim fails then we have

$$|B_1|(\frac{1}{2} - 2\delta) \leq |C^{\varepsilon_j}| = \int_{\delta}^1 f^{\varepsilon_j}(x_1) dx_1 \leq |B_1|(\frac{1}{2} - 3\delta)$$

which is a contradiction.

Hence there is  $x_1^{\varepsilon_j} \in (\delta, 1)$  such that  $f^{\varepsilon_j}(x_1^{\varepsilon_j}) > |B_1|(\frac{1}{2} - 3\delta)$ . Now choose  $0 < a' < a < b < b' < 1$  such that

$$\beta(s) > \kappa, \forall s \in [a', b'].$$

Let  $\varepsilon'_j = \frac{\varepsilon_j}{r}$ . We claim that there is  $\xi^{\varepsilon_j} \in \Lambda^{\varepsilon_j}$  and  $\bar{x}_1^{\varepsilon_j}$  such that

$$\frac{v^{\varepsilon_j}}{\varepsilon'_j}(x_1^{\varepsilon_j}, \xi^{\varepsilon_j}) \in (a, b).$$

Indeed, for sufficiently large  $j$  we have

$$a > \frac{v^{\varepsilon_j}}{\varepsilon'_j}(-x_1^{\varepsilon_j}, x') = 0, \quad \forall x' \in \Lambda^{\varepsilon_j}(x_1^{\varepsilon_j})$$

and

$$\frac{v^{\varepsilon_j}}{\varepsilon'_j}(x_1^{\varepsilon_j}, x') > \frac{\gamma}{2\varepsilon'_j} > b, \quad \forall x' \in \Lambda^{\varepsilon_j}(x_1^{\varepsilon_j})$$

provided that  $j > j_1(r)$ , see Figure 1. Hence from the mean value theorem we see that the claim is true. From the uniform Lipschitz continuity of the functions  $v^{\varepsilon_j}$  it follows that there is a constant  $c_0 > 0$  such that

$$\frac{v^{\varepsilon_j}}{\varepsilon'_j}(x_1, x') \in (a', b'), \quad \text{if } |x_1 - \bar{x}_1^{\varepsilon_j}| < \varepsilon'_j c_0, x' \in \Lambda^{\varepsilon_j}(x_1^{\varepsilon_j}).$$

Consequently we have that

$$\int_{B_1} \beta_{\varepsilon'_j}(v^{\varepsilon_j}) \geq \frac{\kappa}{\varepsilon'_j} \int_{|x_1 - \bar{x}_1^{\varepsilon_j}| < \varepsilon'_j c_0} |\Lambda^{\varepsilon_j}(x_1)| dx_1 \geq \frac{\kappa}{\varepsilon'_j} (1 - 3\delta) 2c_0 \varepsilon'_j = 2\kappa |B_1| (1 - 3\delta) c_0 := \tilde{C}.$$

Now the non-degeneracy follows from the proof of Part II of Theorem 6.3 [CLW97]. The asymptotic expansion follows from Theorem A and Proposition 3.1.  $\square$

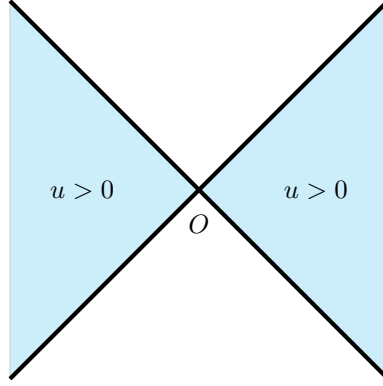


FIGURE 2. Possible self-crossing free boundary which fails to satisfy the conditions of Lemma 5.1.

**Remark 5.2.** Note that under weaker assumption  $\Theta(x_0, \{u_0 > 0\}) > 0$  the argument in the proof of Lemma 5.1 still works. However for the self-crossing free boundary [Wei03] (see Figure 2) the assumptions of Lemma 5.1.

As an immediate corollary we have

**Corollary 5.3.** Let  $x_0 \in \partial\{u_0 > 0\}$  be a point of reduced boundary. Then  $u_0$  is non-degenerate at  $x_0$ .

**Proof.** Suppose that  $0 \in \partial\{u_0 > 0\}$  and  $\partial\{(u_0)_r > 0\} \subset B_2 \cap \{|x \cdot e| < \varepsilon\}$  or some unit vector  $e$  and small  $\varepsilon > 0$ . Here  $(u_0)_r = \frac{u_0(rx)}{r}$ . Consider the family of balls  $B_{1/2}(et), t \in [-\varepsilon, \varepsilon]$ . Then there is  $t_\varepsilon \in [-\varepsilon, \varepsilon]$  such that  $B_{1/2}(et_\varepsilon)$  touches the free boundary at some point  $z_0 \in B_1$  provided that  $\varepsilon$  is sufficiently small. Let  $\nu_0 = t_\varepsilon e$ . Introduce the following barrier function

$$w(x) = \frac{\varphi(1/2) - \varphi(|x - \nu_0|)}{\varphi(1/2) - \varphi(1)} \sup_{B_1(\nu_0)} u_0$$

where  $\varphi(|x|) = \frac{1}{|x|^{N-2}}$ . We have  $\Delta(u_0 - w) = \Delta u_0 \geq 0$  in  $D = B_1(\nu_0) \setminus B_{1/2}(\nu_0)$  and  $u_0 - w \leq 0$  on  $\partial D$ . From the maximum principle we infer that  $u_0 \leq w$  in  $D$ . But we have that the maximum of  $u_0 - w$  is realized at  $z_0$ . Hence from the Hopf lemma we get

$$-\sqrt{2M} = \partial_{\nu_0} u_0(z_0) > \partial_{\nu_0} w(z_0) = -|\varphi'(1/2)| \frac{\sup_{B_1(\nu_0)} u_0}{\varphi(1/2) - \varphi(1)}$$

or

$$\sup_{B_1(\nu_0)} (u_0)_r \geq \sqrt{2M} \frac{\varphi(1/2) - \varphi(1)}{|\varphi'(1/2)|}.$$

□

In the following definition we denote  $\Omega^+(u) = \{u > 0\}$  and  $\Omega^-(u) = \{u < 0\}$ . Moreover, let

$$G(u_\nu^+, u_\nu^-) := (u_\nu^+)^2 - (u_\nu^-)^2 - 2M \quad (5.5)$$

where  $u_\nu^+$  and  $u_\nu^-$  are the normal derivatives in the inward direction  $\nu$  to  $\partial\Omega^+(u)$  and  $\partial\Omega^-(u)$ , respectively. For more details see Definition 2.4 [CS05].

**Definition 5.1.** Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$  and let  $u$  be a continuous function in  $\Omega$ . We say that  $u$  is a viscosity solution in  $\Omega$  if

- i)  $\Delta u = 0$  in  $\Omega^+(u)$  and  $\Omega^-(u)$ ,
- ii) along the free boundary  $\partial\{u > 0\}$ ,  $u$  satisfies the free boundary condition, in the sense that:
  - a) if at  $x_0 \in \partial\{u > 0\}$  there exists a ball  $B \subset \Omega^+(u)$  such that  $x_0 \in \partial B$  and

$$u^+(x) \geq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|), \quad \text{for } x \in B, \quad (5.6)$$

$$u^-(x) \leq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|), \quad \text{for } x \in B^c, \quad (5.7)$$

for some  $\alpha > 0$  and  $\beta \geq 0$ , with equality along every non-tangential domain, then the free boundary condition is satisfied

$$G(\alpha, \beta) \leq 0,$$

- b) if at  $x_0 \in \partial\{u > 0\}$  there exists a ball  $B \subset \Omega^-(u)$  such that  $x_0 \in \partial B$  and

$$u^-(x) \geq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|), \quad \text{for } x \in B,$$

$$u^+(x) \leq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|), \quad \text{for } x \in \partial B,$$

for some  $\alpha \geq 0$  and  $\beta > 0$ , with equality along every non-tangential domain, then

$$G(\alpha, \beta) \geq 0.$$

In our case  $\beta = 0$  and we have only  $u^+$ . However one has to check that the free boundary conditions a) and b) in Definition 5.1 are satisfied.

**Lemma 5.4.** Let  $u_0$  be a blow-up of  $u$  at some non-degenerate point such that  $\Theta(x, \{u_0 > 0\}) \geq \frac{1}{2}$  for every  $x \in \partial\{u_0 > 0\}$ . Then  $u_0$  is a viscosity solution in the sense of Definition 5.1.

**Proof.** We have to show that the properties a), b) in Definition 5.1 hold. Suppose that  $B \subset \{u_0 > 0\}$  touches  $\partial\{u_0 > 0\}$  at some point  $x_0$ . Then it follows from Hopf's lemma that  $u_0$  is non-degenerate at  $x_0$ . Consequently, if  $u_{00}$  is a blow-up at  $x_0$  then by Theorem A  $u_{00}(x) = \alpha x_1^+$  after some rotation of coordinate system. Moreover  $0 < \alpha \leq \sqrt{2M}$ . Hence  $G(\alpha, 0) \leq 0$ .

Now suppose that  $B \subset \{u_0 = 0\}$  and  $B$  touches  $\partial\{u_0 > 0\}$  at  $z_0$ . By Lemma 5.1  $u_0$  is non-degenerate at  $z_0$ . Theorem A implies that any blow-up  $u_{00}$  of  $u_0$  at  $z_0$  must be  $u_{00}(x) = \sqrt{2M}x_1^+$  after some rotation of coordinates. Hence  $G(\sqrt{2M}, 0) \geq 0$ .  $\square$

**5.1. Properties of  $\partial\{u_0 > 0\}$ .** We want to study the properties of  $g$ . We first prove a Bernstein-type result which is a simple consequence of a refinement of Alt-Caffarelli-Friedman monotonicity formula [ACF84a], [CKS00].

**Lemma 5.5.** Let  $u \geq 0$  be a limit of solutions to  $(\mathcal{P}_\varepsilon)$ . Let  $u_0 = rg(\sigma)$ ,  $\sigma \in \mathbb{S}^{N-1}$  be a nontrivial blow-up of  $u$  at some free boundary point. If there is a hemisphere containing  $\text{supp } g$  then the graph of  $u_0$  is a half-plane.



**Proof.** Without loss of generality we assume that  $\text{supp } g \subset \mathbb{S}_+^{N-1} = \{X \in \mathbb{S}^{N-1}, x_N \geq 0\}$ . Let  $v(x_1, \dots, x_N) = u(x_1, \dots, -x_N)$  be the reflection of  $u$  with respect to the hyperplane  $x_N = 0$ . Then  $v$  is nonnegative subharmonic function satisfying the requirements of Lemma 2.3 [CKS00]. Thus

$$\Phi(r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_0|^2}{|x|^{N-2}} \int_{B_r} \frac{|\nabla v|^2}{|x|^{N-2}}$$

is nondecreasing in  $r$ . Moreover

$$\Phi'(r) \geq \frac{2\Phi(r)}{r} A_r, \quad A_r = \frac{C_N}{r^{N-1}} \text{Area}(\partial B_r \setminus (\text{supp } u_0 \cup \text{supp } v)).$$

Thus, if  $\text{supp } g$  digresses from the hemisphere by size  $\delta > 0$  then  $A_r \geq c(\delta) > 0$ . Hence integrating the differential inequality for  $\Phi$  we see that  $\Phi$  grows exponentially which is a contradiction, since in view of Proposition 7.3  $u_0$  is Lipschitz and hence  $\Phi$  must be bounded.  $\square$

It is convenient to define the following subsets of the free boundary

$$\Gamma_{\frac{1}{2}} = \left\{ x \in \partial\{u_0 > 0\} \text{ s.t. } \Theta(x, \{u_0 > 0\}) = \frac{1}{2} \right\}, \quad (5.8)$$

$$\Gamma_1 = \{x \in \partial\{u_0 > 0\} \text{ s.t. } \Theta(x, \{u_0 > 0\}) = 1\}. \quad (5.9)$$

$\Theta(x, D)$  Denotes the Lebesgue density of  $D$  at  $x$ . We will see that  $\Theta(x, \{u_0 > 0\})$  exists at every non-degenerate point and equals either 1 or  $\frac{1}{2}$ .

**Lemma 5.6.** *Assume  $N = 3$ . Let  $x_0 \in \partial\{u_0 > 0\} \setminus \{0\}$  be a non-degenerate free boundary point such that the lower Lebesgue density  $\Theta_*(x_0, \{u_0 \equiv 0\}) > 0$ . Then there is a unit vector  $\nu_0$  such that*

$$u_0(x) = \sqrt{2M}[(x - x_0) \cdot \nu_0]^+ + o(x - x_0). \quad (5.10)$$

In particular,  $x_0 \in \Gamma_{\frac{1}{2}}$

**Proof.** Set  $v_k = \frac{u_0(x_0 + \rho_k x)}{\rho_k}$ , since  $u_0$  is non-degenerate at  $x_0$  it follows from a customary compactness argument that  $v_k \rightarrow v$  and by virtue of Corollary 3.2  $v$  is homogeneous function of degree one. We have

$$\frac{u_0(x_0 + \rho_k x)}{\rho_k} = u_0(\rho_k^{-1} x_0 + x) = \nabla u_0(\rho_k^{-1} x_0 + x)(\rho_k^{-1} x_0 + x) = \quad (5.11)$$

$$= \frac{1}{\rho_k} \nabla u_0(x_0 + \rho_k x)(x_0 + \rho_k x) \quad (5.12)$$

where the last line follows from the zero degree homogeneity of the gradient, hence

$$\rho_k v_k(x) = \nabla u_0(x_0 + \rho_k x)(x_0 + \rho_k x) = \nabla v_k(x)(x_0 + \rho_k x).$$

By Lipschitz continuity of  $u_0$  it follows that  $v_k$  is locally bounded. Consequently, for a suitable subsequence of  $\rho_k$  we have that  $v_{k_j} \rightarrow v$  and  $\nabla v(x)x_0 = 0$ . Without loss of generality we may assume that  $x_0$  is on the  $x_3$  axis, implying that  $v$  depends only on  $x_1$  and  $x_2$ . Applying Proposition 7.6 and Corollary 3.2 we conclude that  $S(x_0, r, u_0)$  is non-decreasing and thus  $v$  must be homogeneous of degree one. Indeed, there is a sequence  $\delta_j \rightarrow 0$  such that  $(u_{\varepsilon_j})_{\lambda_j} \rightarrow v, \delta_j = \varepsilon_j / \lambda_j$  by Proposition 7.6.

Finally, applying Theorem A and the assumption  $\Theta_*(x_0, \{u_0 = 0\}) > 0$  we see that  $v$  must be a half-plane solution. It remains to note that the approximate tangent of  $\partial\{u_0 > 0\}$  at  $x_0$  is unique and this completes the proof.  $\square$

**Lemma 5.7.** *We want to show that  $\Theta(x, \{u_0 > 0\}) \geq \frac{1}{2}$  in some neighbourhood of  $x_0$ . Let  $x_0 \in \Gamma_{\frac{1}{2}}$ . Then there exists  $r_0 > 0$  such that  $B_{r_0}(x_0) \cap \partial\{u_0 > 0\}$  is a  $C^{1,\alpha}$  surface.*

**Proof.** Let  $y_0 \in \partial\{u_0 > 0\}$  be a degenerate point. Suppose there is  $\rho > 0$  such that  $u_0$  is degenerate at every point of  $B_\rho(y_0) \cap \partial\{u_0 > 0\}$ . Since  $\text{supp } \Delta u_0 \subset \partial\{u_0 > 0\}$  then it follows that  $u_0 \equiv 0$  in  $B_\rho(y_0)$ . Consequently, there is a sequence of non-degenerate points  $y_k \rightarrow y_0$ . Note that if  $y_k$  is a non-degenerate point then by Theorem A the Lebesgue density  $\Theta(y_k, \{u_0 > 0\}) \geq \frac{1}{2}$ .

Let  $u_{00}^k$  be a blow-up of  $u_0$  at  $y_k$ . By Proposition 7.6 for fixed  $k$  there are  $\delta_j^k \rightarrow 0$  such that  $(u_{\varepsilon_j^k})_{\lambda_j^k} \rightarrow u_{00}^k$ ,  $\delta_j^k = \varepsilon_j^k / \lambda_j^k$ . Thus applying Theorem A it follows that  $u_{00}^k$  is a half plane solution or a wedge.

From scaling properties of Spruk's monotonicity formula and Lemma 3.4 we get

$$s(0, y_k, u_0) = s(1, 0, u_{00}^k) = 2M \text{vol}(B_1 \cap \{u_{00}^k > 0\}) = \begin{cases} 2\pi M & \text{if } y_k \text{ is a wedge point,} \\ \pi M & \text{otherwise.} \end{cases} \quad (5.13)$$

Then applying Corollary 3.2 to  $u_{\delta_j^k}$  and using the semicontinuity of  $S$  Lemma 3.3 together with Lemma 3.5 we have

$$2M \text{vol}(B_1 \cap \{u_{00}^k > 0\}) = \limsup_{y_k \rightarrow y_0} s(0, y_k, u_0) \leq s(0, y_0, u_0) = 2M\pi\Theta(x_0, \{u_0 > 0\}). \quad (5.14)$$

Therefore we conclude that  $\Theta(x, \{u_0 > 0\}) \geq \frac{1}{2}$  for every free boundary point  $x$  in some neighbourhood of  $x_0$ . By virtue of Lemma 5.4  $u_0$  is a viscosity solution which is flat  $x_0$ . Applying the "flatness-implies- $C^{1,\alpha}$ " regularity results from [Caf87] and [Caf89] the lemma follows.  $\square$

Next we prove a representation formula for  $\Delta u_0$ .

**Lemma 5.8.** *Let  $u_0$  be as in Lemma 5.5. Then*

- (i)  $\mathcal{H}^2(\Gamma_{\frac{1}{2}} \cap B_R) < \infty$ , for any  $R > 0$ ,
- (ii) away from  $\Gamma_1$  the representation formula holds

$$\Delta u_0 = \sqrt{2M} \mathcal{H}^2 \llcorner \Gamma_{\frac{1}{2}}.$$

**Proof.** (i) For given  $x \in \Gamma_{\frac{1}{2}}$  there is a  $\tilde{\rho}_x > 0$  such that

$$\sup_{B_r(x)} u_0 \geq \sqrt{M}r, \quad r \in (0, \tilde{\rho}_x). \quad (5.15)$$

This follows from the asymptotic expansion in Lemma 5.6. Consequently, there is  $\rho'_x > 0$  such that

$$\int_{B_r(x)} \Delta u_0 \geq \sqrt{M}r^2, \quad r \in (0, \rho'_x). \quad (5.16)$$

Indeed, if this inequality is false then there is a sequence  $r_j \searrow 0$  such that

$$\int_{B_{r_j}(x)} \Delta u_0 < \sqrt{M} r_j^2.$$

Set  $v_j(x) = \frac{u_0(x+r_j x)}{r_j}$ . By (5.15)  $\sup_{B_1} v_j(x) \geq \sqrt{M}$ . Moreover, it follows from Lemma 5.6 that  $v_j(x) \rightarrow \sqrt{2M} x_1^+$  in a suitable coordinate system, while  $\int_{B_1} \Delta v \leq \sqrt{M}$ . However,  $\int_{B_1} \Delta x_1^+ = \sqrt{2M} \frac{\pi}{2}$  and this is in contradiction with the former inequality. Putting  $\bar{\rho}_x = \min(\rho'_x, \tilde{\rho}_x)$  we see that the collection of balls  $\mathcal{F} = \bigcup B_{\rho_x}(x), x \in \Gamma_{\frac{1}{2}} \cap B_R, \rho_x < \bar{\rho}_x$  is a Besicovitch type covering of  $\Gamma_{\frac{1}{2}} \cap B_R$ . Consequently, there is a positive integer  $m > 0$  and subcoverings  $\mathcal{F}_1, \dots, \mathcal{F}_m$  such that the balls in each of  $\mathcal{F}_i, 1 \leq i \leq m$  are disjoint and  $\Gamma_{\frac{1}{2}} \cap B_R \subset \bigcup_{i=1}^m \mathcal{F}_i$ . We have from (5.16)

$$4\pi R^2 \|\nabla u_0\|_\infty \geq \int_{\partial B_R} \partial_\nu u_0 \geq \int_{B_{\rho_x}(x) \in \mathcal{F}_i} \Delta u_0 = \sum_{B_{\rho_x}(x) \in \mathcal{F}_i} \int_{B_{\rho_x}(x)} \Delta u_0 \geq m\sqrt{M} \sum_{B_{\rho_x}(x) \in \mathcal{F}_i} \rho_x^2.$$

This yields

$$\sum_{B_{\rho_x}(x) \in \bigcup_{i=1}^m \mathcal{F}_i} \rho_x^2 \leq \frac{4\|\nabla u_0\|_\infty \pi R^2}{m\sqrt{M}}. \quad (5.17)$$

Given  $\delta > 0$  small, suppose there is  $x \in \Gamma_{\frac{1}{2}}$  such that  $\bar{\rho}_x \geq \delta$ . Then we choose  $\rho_x < \delta$ . Thus, in any case we can assume that  $\rho_x < \delta$ . In view of (5.17) this implies that the  $\delta$ -Hausdorff premeasure is bounded independently of  $\delta$ . This proves (i).

(ii) From the estimate

$$\sqrt{M} r^2 \leq \int_{B_r(x)} \Delta u_0 \leq 4\pi r^2 \|\nabla u_0\|, \quad r \in (0, \bar{\rho}_x), B_r(x) \cap \Gamma_{\frac{1}{2}} \subset \Gamma_{\frac{1}{2}}$$

we see that there is a positive bounded function  $q$  such that  $\Delta u_0 = q \mathcal{H}^2 \llcorner \Gamma_{\frac{1}{2}}$ . Using Lemma 5.6 we conclude that  $q = \sqrt{2M}$ .  $\square$

Next we prove the full non-degeneracy of  $u_0$  near  $\Gamma_{\frac{1}{2}}$ .

**Lemma 5.9.** *Let  $u_0$  be as above and  $x_0 \in \Gamma_{\frac{1}{2}}$  then for any  $B_r(x)$  such that  $x \in \partial\{u_0 > 0\}, B_r(x) \cap \partial\{u_0 > 0\} \subset \Gamma_{\frac{1}{2}}$  we have*

$$\sup_{B_r(x)} u_0 \geq \sqrt{2M} \pi r.$$

**Proof.** By a direct computation we have

$$r^{-2} \int_{\partial B_r(x)} u_0 = \int_0^r \frac{dt}{t^2} \int_{B_t(x)} \Delta u_0 \geq \int_0^r \frac{1}{t^2} \sqrt{2M} \pi t^2 = \sqrt{2M} \pi r$$

where the inequality follows from the representation formula and the fact that  $\partial\{u_0 > 0\}$  is a cone, and hence for all  $t \in (0, r)$   $\mathcal{H}^2(B_t(x) \cap \Gamma_{\frac{1}{2}}) \geq \pi t^2$ . It remains to note that  $r^{-2} \int_{\partial B_r(x)} u_0 \leq \sup_{B_r(x)} u_0$ .  $\square$

**5.2. Weak solutions.** Combining Lemmas 5.8 and 5.9 as well as Propositions 7.2 (iii) and 7.3 (i) we see that  $u_0$  is a weak solution near  $\Gamma_{\frac{1}{2}}$  in the sense of Definition 5.1 [AC81]. Furthermore,  $\partial\{u_0 > 0\} \setminus \{0\}$  is flat at each point.

**Lemma 5.10.**  $u_0$  is a weak solution in Alt-Caffarelli sense away from  $\Gamma_1$ . Furthermore,  $\Gamma_{\frac{1}{2}}$  is smooth.

**Proof.** All conditions in Definition 5.1 [AC81] are satisfied and  $u_0$  is flat at every point  $z_0 \in \partial\{u_0 > 0\} \setminus \{0\}$  thanks to (5.10). Applying Theorem 8.1 [AC81] we infer that  $\Gamma_{\frac{1}{2}}$  is smooth at every  $z_0 \in \partial\{u_0 > 0\} \setminus \{0\}$ .  $\square$

**5.3. Minimal perimeter.** In this section we prove that the local perturbations  $S' \subset \{u_0 > 0\}$  of a portion  $S \subset \Gamma_{\frac{1}{2}}$  has larger  $\mathcal{H}^2$  measure than  $S$ . This can be seen from the estimate  $|\nabla u_0(x)| \leq \sqrt{2M}$  which follows from Lemma 7.7. Since by Lemma 5.10 on  $\Gamma_{\frac{1}{2}}$  the free boundary condition  $|\nabla u_0| = \sqrt{2M}$  is satisfied in the classical sense, it follows that

$$\begin{aligned} 0 &= \int_D \Delta u_0 = \int_S \partial_\nu u_0 + \int_{S'} \partial_\nu u_0 = \\ &= \sqrt{2M} \mathcal{H}^2(S) + \int_{S'} \partial_\nu u_0, \end{aligned}$$

where  $D \subset \{u_0 > 0\}$  such that  $\partial D = S \cup S'$ . But  $|\int_{S'} \partial_\nu u_0| \leq \sqrt{2M} \mathcal{H}^2(S')$  and thereby

$$\mathcal{H}^2(S) \leq \mathcal{H}^2(S'). \quad (5.18)$$

The estimate for the perimeter can be reformulated as follows:

**Theorem 5.11.** Let  $N = 3$ , then the components of  $\Gamma_{\frac{1}{2}}$  are surfaces of non-positive outward mean curvature. In particular,  $\Gamma_{\frac{1}{2}}$  is a union of smooth convex surfaces.

**Proof.** Since  $u_0$  is a weak solution then by Lemma 5.10  $\Gamma_{\frac{1}{2}}$  is smooth. If  $z_0 \in \Gamma_{\frac{1}{2}}$  then choosing the coordinate system in  $\mathbb{R}^3$  so that  $x_3$ -axis has the direction of the inward normal of  $\{u_0 > 0\}$  at  $z_0$  and considering the free boundary near  $z_0$  as a graph  $x_3 = h(x_1, x_2)$  we can consider the one-sided variations of the surface area functional. Indeed, let  $\mathcal{D} \subset \mathbb{R}^2$  be a open bounded domain in  $x_1 x_2$  plane containing  $z_0$  and assume  $t > 0, 0 \leq \psi \in C_0^\infty(\mathcal{D})$ . Then from (5.18) we have

$$\begin{aligned} 0 &\geq \frac{1}{t} \int_{\mathcal{D}} [\sqrt{1 + |\nabla h|^2} - \sqrt{1 + |\nabla(h - t\psi)|^2}] = \\ &= \int_{\mathcal{D}} \frac{2\nabla h \nabla \psi - t|\nabla \psi|^2}{\sqrt{1 + |\nabla h|^2} + \sqrt{1 + |\nabla(h - t\psi)|^2}} \rightarrow \text{ as } t \rightarrow 0 \\ &\rightarrow \int_{\mathcal{D}} \frac{\nabla h \nabla \psi}{\sqrt{1 + |\nabla h|^2}}. \end{aligned} \quad (5.19)$$

Therefore  $\operatorname{div} \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \geq 0$  and noting that  $\Gamma_{\frac{1}{2}}$  is a cone the result follows.  $\square$

#### 5.4. Full Non-degeneracy.

**Lemma 5.12.** Assume that  $N = 3$  and let  $u_0$  be a nontrivial blow-up of  $u$  such that the measure theoretic boundary of  $\{u_0 > 0\}$  is non-empty. Then  $\partial\{u_0 > 0\} \setminus \{0\} \subset \Gamma_{\frac{1}{2}}$ . In particular the set of degenerate points of  $\partial\{u_0 > 0\}$  is empty.

**Proof.** Let  $u_0$  be a blow-up of  $u$  at 0. Since  $u$  is non-degenerate at 0 then it follows that  $u_0$  does not vanish identically. Hence there is a ball  $B \subset \{u_0 > 0\}$  touching  $\partial\{u_0 > 0\}$  at some point  $z_0 \in \partial\{u_0 > 0\} \cap B$ . By Hopf's lemma, Lipschitz estimate 7.3 (i) and asymptotic expansion [Caf89] Lemma A1 it follows that  $u_0$  is not degenerate at  $z_0$ . Consequently, the set of non-degenerate points of  $u_0$  is not empty.

Suppose that  $S$  is a component of  $\partial\{u_0 > 0\}$  containing a point of measure theoretic boundary of  $\{u_0 > 0\}$ . Note that by Lemma 5.7 and Theorem 5.11  $S$  is a smooth convex surface. Let  $x_0 \in \partial S, x_0 \neq 0$ . Then either a)  $x_0 \in \Gamma_1$  or b)  $u_0$  is degenerate at  $x_0$ .

We first analyze the case a). Let  $\ell$  be the ray passing through  $x_0$  and  $\Pi$  the tangent half-plane to  $S$  along  $\ell$ . First note that  $u_0$  is non-degenerate at  $x_0$  because

$$\int_{B_r(x_0)} \Delta u_0 \geq \int_{B_r(x_0) \cap S} \Delta u_0 \geq \sqrt{2M} \mathcal{H}^2(S \cap B_r(x_0)) \geq \sqrt{2M} \frac{\pi r^2}{2}$$

for sufficiently small  $r$ . Consequently

$$\frac{1}{R^2} \int_{\partial B_R} u_0 = \int_0^R \frac{1}{r^2} \int_{B_r(x_0)} \Delta u_0 \geq \sqrt{2M} \frac{\pi r^2}{2} R. \quad (5.20)$$

Let  $u_{00}$  be a blow-up  $u_0$  at  $x_0$ . Then from Theorem A it follows that  $u_{00}$  is two dimensional. Moreover  $\Pi \subset \partial\{u_{00} > 0\}$ ,  $\{u_{00} > 0\}$  has unit density at 0, and the interior of  $\{u_{00} = 0\}$  near  $\Pi$  is not empty. Note that the interior of the set  $\{u_0 = 0\}$  propagates to  $x_0$  along another component  $S_1$  of measure theoretic boundary, see Figure 3. Consequently, near  $\Pi$   $u_{00}(x) = \sqrt{2M}x_1^+$  after some rotation of coordinates. From the unique continuation theorem it follows that  $u_{00}(x) = \sqrt{2M}x_1^+$  everywhere which is in contradiction with the fact that  $\{u_{00} > 0\}$  has unit density at 0.

As for the case b) then (5.20) shows that  $u_0$  is non-degenerate at  $x_0$  as long as  $x_0$  is on the boundary of  $S$ . □

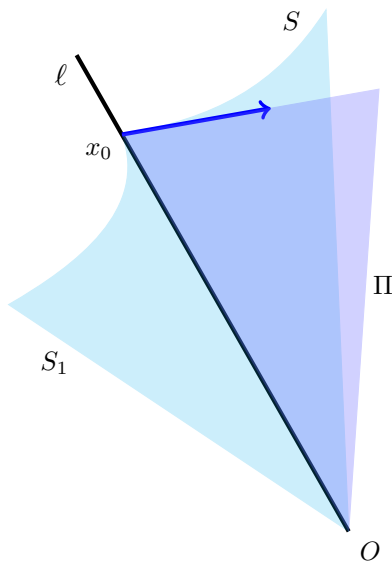
## 5.5. Properties of $\Gamma_{\frac{1}{2}}$ .

**Lemma 5.13.** *Suppose  $u_0$  is not degenerate at  $x_0 \in \partial\{u_0 > 0\} \setminus \{0\}$ , such that  $\Theta_*(x_0, \{u_0 = 0\}) > 0$ . Then there is a unique component  $\mathfrak{C}$  of  $\partial\{u_0 > 0\}$  containing  $x_0$  such that  $\mathfrak{C} \subset \Gamma_{\frac{1}{2}}$ .*

**Proof.** We only have to show the uniqueness of  $\mathfrak{C}$ , the rest follows from Lemmata 5.6 and 5.7. Suppose, there are two components of  $\partial\{u_0 > 0\} \setminus \{0\}$ ,  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ , containing  $x_0$ . From the dimension reduction argument as in the proof of Lemma 5.6 it follows that  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  have the same approximate tangent plane at  $x_0$ . This is in contradiction with our assumption  $\Theta_*(x_0, \{u_0 = 0\}) > 0$ . □

**Lemma 5.14.** *Let  $\mathfrak{C}$  be a component of  $\partial\{u_0 > 0\}$  such that  $\mathfrak{C} \cap \Gamma_{\frac{1}{2}} \neq \emptyset$ . Then  $\mathfrak{C} \setminus \Gamma_{\frac{1}{2}} = \emptyset$ , in other words all points of  $\mathfrak{C}$  are in  $\Gamma_{\frac{1}{2}}$ .*

**Proof.** By Lemma 5.12  $\mathfrak{C}$  cannot have degenerate points, thus we have to show that  $\Gamma_{\frac{1}{2}}$  cannot have limit points in  $\Gamma_1$ . Note that  $\Gamma_{\frac{1}{2}}$  is of locally finite perimeter (see Lemma 5.8 (i)) and hence locally it is a countable union of convex surfaces. Let  $x_0 \in \Gamma_1 \cap \mathfrak{C}$  be a limit point of  $\Gamma_{\frac{1}{2}} \cap \mathfrak{C}$ . The generatrix of the

FIGURE 3. The structure of free boundary near the point  $x_0$ .

cone  $\partial\{u_0 > 0\}$  passing through  $x_0$  splits  $\mathfrak{C}$  into two parts one of which must be convex near  $x_0$  because by assumption  $x_0$  is a limit point of  $\Gamma_{\frac{1}{2}}$ , see Theorem 5.11. The set  $\{u_0 = 0\}^\circ$  propagates to  $x_0$  because  $\Gamma_{\frac{1}{2}}$  is a subset of reduced boundary. Thus, there is another subset of  $\Gamma_{\frac{1}{2}}$  approaching to  $x_0$ , and it is a part of the topological boundary of  $\{u_0 = 0\}^\circ$ . Therefore, the ray passing through  $x_0$  is on the boundaries of two convex pieces of  $\partial\{u_0 > 0\}$  (near  $x_0$ ). Note that if these pieces of  $\Gamma_{\frac{1}{2}}$  contain flat parts then from the unique continuation theorem we infer that  $\partial\{u_0 > 0\}$  cannot have singularity at 0. Thus, they cannot contain flat parts and consequently the density of  $\{u_0 > 0\}$  at  $x_0$  cannot be 1, because by convexity of  $\Gamma_{\frac{1}{2}}$  it follows that  $\{u_0 \equiv 0\}^\circ$  has positive density at  $x_0$ . But this is in contradiction with the assumption  $x_0 \in \Gamma_1$ .  $\square$

Summarizing we have

**Proposition 5.15.** *Let  $u_0$  be as above and  $N = 3$ , then  $\partial\{u_0 > 0\} \setminus \{0\}$  is a union of smooth convex cones.*

**5.6. Proof of Theorem B.** The first part of Theorem B follows from Lemma 5.12 while the second part is a corollary of Lemma 5.14 since  $\Gamma_{\frac{1}{2}}$  coincides with the reduced boundary. Finally, the last part follows from Lemma 5.10, because by Lemma 5.14 the reduced boundary propagates instantaneously in  $\partial\{u_0 > 0\}$ .

## 6. PROOF OF THEOREM C

**6.1. Inverse Gauss map and the support function.** Suppose  $N = 3$  and  $u = rg(\theta, \phi)$ , where

$$x = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

then

$$\Delta u_0 = \frac{1}{r} \left( g_{\theta\theta} + \frac{\cos \theta}{\sin \theta} g_\theta + \frac{g_{\phi\phi}}{\sin^2 \theta} + 2g \right).$$

Note that

$$\Delta_{\mathbb{S}^2} g = g_{\theta\theta} + \frac{\cos \theta}{\sin \theta} g_\theta + \frac{g_{\phi\phi}}{\sin^2 \theta}$$

is the Laplace-Beltrami operator. Thus we get

$$\Delta_{\mathbb{S}^2} g + 2g = 0. \quad (6.1)$$

Let  $H(n), n \in \mathbb{S}^{N-1}$  be the Minkowski support function of some hypersurface  $\mathcal{M}$ .  $H(n)$  is the distance between the point on  $\mathcal{M}$  with normal  $n$  and the origin. It is known [Ale39] that the eigenvalues of the matrix

$$\nabla_{ij}^2 H(n) + \delta_{ij} H(n)$$

are the principal radii of curvature of the surface determined by  $H$ , where the second order derivatives are taken with respect to an orthonormal frame at  $n \in \mathbb{S}^{N-1}$ . The support function uses the inverse of the Gauss map to parametrize the surface as follows

$$H(n) = G^{-1}(n) \cdot n.$$

Furthermore, we have the following formula for the Gauss curvature  $K$  [Ale39]

$$\frac{1}{K} = \det(\nabla_{ij}^2 H(n) + \delta_{ij} H(n)). \quad (6.2)$$

The Gauss map is a local diffeomorphism whenever  $K \neq 0$  [LR88]. Since  $u_0 = rg$  is harmonic in  $\{u_0 > 0\}$  we infer that  $g$  is smooth on  $\mathbb{S}^2 \cap \{g > 0\}$ .

**Remark 6.1.** *In higher dimensions the equation (6.1) becomes*

$$\Delta_{\mathbb{S}^{N-1}} g + (N-1)g = \sum_{i=1}^{N-1} \frac{1}{k_i} = \frac{\sigma_{N-2}(k)}{\sigma_{N-1}(k)} = 0, \quad \sigma_m = \sum_{i_1 < i_2 < \dots < i_m} k_{i_1} k_{i_2} \dots k_{i_m}, \quad (6.3)$$

where  $\sigma_m(k)$  is the  $m$ -th elementary symmetric function and  $k_i, i = 1, \dots, N-1$  are the principal curvatures. Observe that any positive function  $g > 0$  satisfying the equation  $\Delta_{\mathbb{S}^{N-1}} g + (N-1)g = 0$  defines a  $(N-2)$ -minimal surface (i.e.  $\sigma_{N-2}(k) = 0$ ) provided that the Gauss curvature  $\sigma_{N-1} \neq 0$ . From here we infer that the spherical parts of homogeneous stationary points of the Alt-Caffarelli functional are support function of  $(N-2)$ -capillary surface in  $\mathbb{S}^{N-1}$ , because they are solutions to (6.3).

**6.2. Catenoid is a solution.** In [AC81] page 110 Alt and Caffarelli constructed a weak solution which is not a minimizer. Their solution can be given explicitly as follows: let

$$x = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

and take

$$u(x) = r \max \left( \frac{f(\theta)}{f'(\theta_0)}, 0 \right)$$

where

$$f(\theta) = 2 + \cos \theta \log \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) = 2 + \cos \theta \log \left( \tan^2 \frac{\theta}{2} \right)$$

and  $\theta_0$  is the unique zero of  $f$  between 0 and  $\frac{\pi}{2}$ . The aim of this section is to show that  $f$  is the support function of catenoid. Recall that the principal radii of curvature of a smooth surface are the eigenvalues of the matrix  $\nabla_{\mathbb{S}^{N-1}}^2 H + \delta_{ij} H$  where the Hessian is taken with respect to the sphere  $\mathbb{S}^{N-1}$  [Ale39]. At each point where the Gauss curvature does not vanish the zero mean curvature condition for  $N = 3$  can be written as

$$\Delta_{\mathbb{S}^2} H + 2H = 0$$

where  $\Delta_{\mathbb{S}^2}$  is the Laplace-Beltrami operator and  $H(n)$  is the value of Minkowski's support function corresponding to the normal  $n \in \mathbb{S}^2$ . From now on let us consider the  $(x, y)$  variables on  $\mathbb{R}^2$ . Recall that by rotating the graph of  $y(x) = a \cosh \frac{x}{a}$  around the  $x$ -axis one obtains a catenoid for some constant  $a$ . Thus it is enough to compute the support function for the graph of  $y$ . Let  $\alpha$  be the angle the tangent line of  $y$  at  $(x, y(x))$  forms with the  $x$ -axis. If  $n$  is the unit normal to the graph of  $y$  then  $n = (-\sin \alpha, \cos \alpha)$  and

$$H(n) = (x, y(x)) \cdot n = -x \sin \alpha + a \cos \alpha \cosh \frac{x}{a}.$$

Noting that the unit tangent at  $(x, y(x))$  is  $(\cos \alpha, \sin \alpha)$  and equating with the slope of tangent line, which is  $(\sinh \frac{x}{a}, -1)$ , we obtain

$$\cos \alpha = \frac{\sinh \frac{x}{a}}{\sqrt{1 + \sinh^2 \frac{x}{a}}}, \quad \sin \alpha = -\frac{1}{\sqrt{1 + \sinh^2 \frac{x}{a}}}.$$

From second equation we get that  $\sinh \frac{x}{a} = \tan \alpha$  and solving the quadratic equation  $e^{2\frac{x}{a}} - 1 = 2e^{\frac{x}{a}} \tan \alpha$  we find that

$$x = a \log \frac{1 + \sin \alpha}{\cos \alpha}, \quad \cosh \frac{x}{a} = \frac{1}{\cos \alpha}.$$

Consequently,

$$H(n) = -\frac{a}{2} \sin \alpha \log \left( \frac{1 + \sin \alpha}{\cos \alpha} \right)^2 + a.$$

Taking  $\alpha = \theta + \frac{\pi}{2}$  we have

$$\frac{1 + \sin \alpha}{\cos \alpha} = \frac{1 + \cos \theta}{-\sin \theta} = \frac{2 \cos^2 \frac{\theta}{2}}{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = -\cot \frac{\theta}{2}$$

and thus choosing  $a = 2$  the result follows.

**6.3. Almost minimal immersions.** Consider the parametrization  $\mathcal{X} : U_g \rightarrow \mathbb{R}^3$ , where

$$\mathcal{X}(n) = ng(n) + \nabla_{\mathbb{S}^2} g, \quad U_g = \{g > 0\} \subset \mathbb{S}^2. \quad (6.4)$$

Let  $\mathcal{M}$  be the hypersurface determined by  $\mathcal{X}$ . The spherical part  $g$  of  $u_0$  solves the equation (6.1) and by Theorem 1 [Rez92]  $\mathcal{X}$  determines a smooth map which is either constant or a conformal minimal immersion outside locally finite set of isolated singularities (branch points). Recall that if at some point  $p$

$$\mathcal{X}_{\xi_1} \times \mathcal{X}_{\xi_2} = 0, \mathcal{X} = \mathcal{X}(\xi_1, \xi_2) \text{ in local coordinates } \xi_1, \xi_2, \quad (6.5)$$



then  $p$  is called branch point, see [Nit89] page 314.

Observe that  $\mathcal{X}(n)$  is the gradient of the blow-up  $u_0$  at  $n = \frac{x}{|x|}$ . Indeed,

$$\begin{aligned}
 \mathcal{X}(n) &= \frac{n}{r}rg + \frac{1}{r}\nabla_{\mathbb{S}^2}(rg) = \\
 &= \frac{n}{r}u_0(x) + \frac{1}{r}\nabla_{\mathbb{S}^2}u_0 = \\
 &= \frac{n}{r}(\nabla u_0(x) \cdot x) + \frac{1}{r}\nabla_{\mathbb{S}^2}u_0 = \\
 &= n(\nabla u_0(x) \cdot \frac{x}{|x|}) + \frac{1}{r}\nabla_{\mathbb{S}^2}u_0 = \\
 &= \nabla u_0(x).
 \end{aligned} \tag{6.6}$$

In particular, the computation above shows that

$$\nabla u_0(x) = \nabla u_0\left(\frac{x}{|x|}\right), \quad \nabla_{\mathbb{S}^2}g(n) \perp n, \tag{6.7}$$

in other words the gradient is homogeneous of degree zero.

The absence of branch points does not rule out the possibility of self-intersection. Therefore we need to prove that under conditions of Theorem C  $\mathcal{M}$  is embedded.

**6.4. Dual cones and center of mass.** If  $u_0$  is a blow-up and the assumptions in Theorem C are satisfied, then by virtue of Proposition 5.15 the free boundary  $\partial\{u_0 > 0\} \setminus \{0\}$  is a union of smooth convex cones  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . We define the dual cones as follows

$$\mathcal{C}_i^* = \partial\{y \in \mathbb{R}^3 : x \cdot y \leq 0, x \in \mathcal{C}_i\}, \quad i = 1, 2. \tag{6.8}$$

It is well-known that the dual of a convex cone is also convex, [Sch14] page 35.

**Lemma 6.2.** *The largest principal curvature of  $\mathcal{C}_i \setminus \{0\}$  is strictly positive.*

**Proof.** To fix the ideas we prove the statement for  $\mathcal{C}_1$ . Note that one of the principal curvatures of  $\mathcal{C}_1 \setminus \{0\}$  is zero because  $\mathcal{C}_1$  is a cone and  $\mathcal{C}_1 \setminus \{0\}$  is smooth, see Theorem B. Let  $\kappa(p)$  be the largest principal curvature at  $p \in \mathcal{C}_1 \setminus \{0\}$ . Suppose there is  $p$  such that  $\kappa(p) = 0$ . Choose the coordinate system at  $p$  so that  $x_1$  points in the outward normal direction at  $p$  (into  $\{u_0 \equiv 0\}$ ),  $x_2$  axis is tangential at  $p$  and is the principal direction corresponding to  $\kappa(p)$ . Then we have that  $\nabla u_0(p) = e_1$ , the unit direction of  $x_1$  axis and the mean curvature of  $\mathcal{C}_1$  at  $p$  vanishes because we assumed that  $\kappa(p) = 0$ . Writing the mean curvature at  $p$  in terms of the derivatives of  $u_0$  we have

$$0 = \frac{\nabla u_0 D^2 u_0 (\nabla u_0)^T - |\nabla u_0|^2 \Delta u_0}{|\nabla u_0|^3} = \frac{\partial_{11} u_0}{\sqrt{2M}}$$

implying that  $\partial_{11} u_0 = 0$ . Moreover, since  $u_0$  is homogeneous of degree one then  $\nabla u_0 = e_1$  along the  $x_1$  axis. This yields  $\partial_{13} u_0 = \partial_{23} u_0 = \partial_{33} u_0 = 0$  along the  $x_1$  axis. From the harmonicity of  $u_0$  it follows that  $\partial_{22} u_0 = 0$  along the  $x_3$  axis. Summarizing, we have that along the points of the  $x_3$  axis the Hessian of  $u_0$

has the following form

$$\begin{pmatrix} 0 & \partial_{12}u_0 & 0 \\ \partial_{12}u_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, letting  $\sigma(t), t \in (-\delta, \delta)$  be the parametrization of the curve along which the  $x_1x_2$  plane intersects with  $\mathcal{C}_1$  and differentiating  $|\nabla u_0(\sigma(t))| = 1$  in  $t$  we get that at  $p$  one must have

$$0 = e_1 \begin{pmatrix} 0 & \partial_{12}u_0 & 0 \\ \partial_{12}u_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e_2 = \partial_{12}u_0(p).$$

Thus, the Hessian  $D^2u_0$  vanishes along the  $x_1$  axis. The function  $w = \sqrt{2M} - \partial_1u_0$  is harmonic in  $\{u_0 > 0\}$  and  $w \geq 0$  thanks to Lemma 7.7. Moreover,  $w(e_1) = 0 = \min w$ . Since at  $e_1$  the free boundary is regular then by Hopf's lemma  $\partial_1w = -\partial_{11}u_0 \neq 0$ . However  $D^2u_0(te_1) = 0$  for every  $t > 0$  and hence  $\partial_{11}w(e_1) = 0$  which is a contradiction.  $\square$

**Remark 6.3.** It follows from Lemma 6.2 and Theorem B that there are two positive constants  $\kappa_0, \kappa_1$  such that

$$0 < \kappa_0 \leq \kappa(p) \leq \kappa_1, \quad p \in (\partial\{u_0 > 0\} \setminus \{0\}) \cap \partial B_{\sqrt{2M}},$$

where  $\kappa(p)$  is the largest curvature of  $\partial\{u_0 > 0\}$  at  $p \in (\partial\{u_0 > 0\} \setminus \{0\}) \cap \partial B_{\sqrt{2M}}$ .

Let us put  $\gamma_i = \mathbb{S}^2 \cap \mathcal{C}_i^*$ .

**Lemma 6.4.** Let  $\mathcal{C}_1^*, \mathcal{C}_2^*$  be the dual cones (6.8). Then we have

- (i)  $\partial\mathcal{M}$  is differentiable and there are two positive constants  $\kappa_0^*, \kappa_1^*$  such that the largest curvature  $\kappa^*(p)$  of  $(\mathcal{C}_i^* \setminus \{0\}) \cap \mathbb{S}^2$  satisfies  $\kappa_0^* \leq \kappa^*(p) \leq \kappa_1^*$ ,
- (ii) there is  $\delta > 0$  small such that every component  $E_\delta$  of  $\partial B_{1-\delta} \cap \mathcal{M}$  defines a convex cone  $K_\delta = \{\sigma t : \sigma \in E_\delta, t > 0\}$ ,
- (iii)  $\mathcal{M}$  is star-shaped with respect to the origin and hence embedded.

**Proof.** Suppose that  $\mathcal{C}_1^*$  is not differentiable at some  $z \neq 0$ . Then  $\mathcal{C}_i$  must have a flat piece. Indeed, if  $n_1, n_2$  are the normals of two supporting hyperplanes of  $\mathcal{C}_i^*$  at  $z$  then the unit vectors  $n_t = \frac{tn_1 + (1-t)n_2}{|tn_1 + (1-t)n_2|}$ , define a support function at  $z$  for every  $t \in (0, 1)$ . Since the vectors  $n_t$  lie on the same plane then  $\mathcal{C}_1$  must have a flat piece. The unique continuation theorem implies that the free boundary is a hyperplane and cannot have singularities. Now the desired estimate follows from Remark 6.3 and the definition of dual cone. The first claim is proved.

Let  $k_1, k_2$  be the principal curvatures of  $\mathcal{M}$ , then  $k_1 + k_2 = 0$  and the Gauss curvature is  $K = -k_1^2 = -k_2^2$ . Since  $\mathcal{M}$  is a smooth immersion then from (6.2) and the smoothness of  $\mathcal{X} = \nabla u_0$  in  $U_g$  we see that  $K \neq 0$ . Furthermore, there is a tame constant  $c_0 > 0$  such that  $k_i^2 \geq c_0, i = 1, 2$  at every point of  $\mathcal{M}$ . Thus by virtue of the part (i)  $\mathcal{M}$  is fibred by  $\partial B_{1-\delta}$  for  $\delta > 0$  small. We claim that  $|\mathcal{X}(n)| > 0, n \in \overline{U_g}$ . Clearly this is true if  $n \in \partial U_g$  where  $|\mathcal{X}(n)| = 1$ . Suppose there is  $n \in U_g$  such that  $\mathcal{X}(n) = 0$ . Since  $\mathcal{X}(n) = ng + \nabla_{\mathbb{S}^2}g$  it follows that  $g(n) = 0$ , but this is impossible since  $n \in \{g > 0\} = U_g$ . From  $g(n) = \mathcal{X}(n) \cdot n > 0, n \in U_g$

it follows that  $\mathcal{M}$  is starshaped with respect to the origin. Consequently,  $\mathcal{M}$  is fibered by  $\partial B_t$  for every  $t \in (0, 1)$  and hence embedded.  $\square$

Let  $n \in U_g$  then from  $\mathcal{X}(n) = \nabla u_0(n)$  it follows that

$$|\mathcal{X}(n)|_{\partial\{u_0>0\}} = |\nabla u_0|_{\partial\{u_0>0\}} = \sqrt{2M}.$$

Since by Lemma 6.4  $\mathcal{M}$  is differentiable along  $\gamma_i$  we see that the contact angle  $\alpha$  between  $\mathcal{M}$  and  $\mathbb{S}^2$  is

$$\cos \alpha = n \cdot \frac{\mathcal{X}(n)|_{\partial\{u_0>0\}}}{\sqrt{2M}} = g(n)|_{\partial\{u_0>0\}} = 0.$$

Thus, the minimal surface defined by  $g$  is inside of the sphere of radius  $\sqrt{2M}$  because in view of Lemma 7.7  $|\nabla u_0|^2 = g^2 + |\nabla g|^2 \leq 2M$ . Moreover,  $\mathcal{M}$  is tangential to  $\mathcal{C}_1^*$  and  $\mathcal{C}_2^*$  along  $\mathbb{S}^2$  since  $n \perp \nabla_{\mathbb{S}^2} g$  by (6.7).

We recall the definition of topological type  $[\varepsilon, r, \chi]$  of hypersurface  $\mathcal{M} \subset \mathbb{R}^3$  from [Nit85] page 47.

**Definition 6.1.** *We say that  $\mathcal{M}$  is of topological type  $[\varepsilon, r, \chi]$  if it has orientation  $\varepsilon$ , Euler characteristic  $\chi$ , and  $r$  boundary curves. Here  $\varepsilon = \pm 1$ , where  $+1$  means that  $\mathcal{M}$  is orientable and  $\varepsilon = -1$  non-orientable. For orientable surface the Euler characteristic is defined by the relation  $\chi = 2 - 2g - r$  where  $g$  is the genus of  $\mathcal{M}$ .*

Now the first part of Theorem C follows from Nitsche's theorem, see page 2 [Nit85]. Moreover, the only stationary surfaces of disk type are the totally geodesic disks and the spherical cups. From Lemma 5.5 it follows that if  $u_0 = rg$  and  $\text{supp } g$  is a disk then  $u_0$  is a half plane.

In view of Lemma 6.4 (iii) the proof of Theorem C can be deduced from the result of Nitsche [Nit62] but we will sketch a shorter proof based on Aleksandrov's moving plane method and Serrin's boundary lemma. We reformulate Theorem C as follows

**Lemma 6.5.** *Let  $\mathcal{M}$  be of topological type  $[1, 2, 0]$ , i.e. a ring-type minimal surface. Then  $\mathcal{M}$  is a part of catenoid.*

**Proof.** By Lemma 6.4 (iii)  $\mathcal{M}$  is embedded. In particular,  $\mathcal{X}$  is a conformal minimal immersion (see the discussion in Section 6.3).

Let  $\partial\mathcal{M} = \gamma_1 \cup \gamma_2$ . Then applying Stokes formula we have

$$\int_{\mathcal{M}} \Delta_{\mathcal{M}} \mathcal{X} = \int_{\partial\mathcal{M}} n^* ds = \int_{\gamma_1} n^* + \int_{\gamma_2} n^* ds \quad (6.9)$$

where  $n^*$  is the outward conormal, i.e.  $n^*$  is tangent to  $\mathcal{M}$  and normal to  $\partial\mathcal{M}$ , see [Fan96] page 81. Since  $\mathcal{X}$  is minimal then  $\Delta_{\mathcal{M}} \mathcal{X} = 0$ . Thus

$$\int_{\gamma_1} n^* ds + \int_{\gamma_2} n^* ds = 0. \quad (6.10)$$

Since  $\mathcal{M}$  is tangential to  $\mathcal{C}_i^*$  it follows that the conormal  $n^*$  on  $\gamma_i$  points in the direction of the generatrix of the dual cone  $\mathcal{C}_i^*$ . Observe that if we use the arc-length parametrization of  $\gamma_i$  and let  $s_k \in [0, |\gamma_i|]$  be some partition points then the sums  $S_m = \sum_{k=0}^m n_k^{*i}(s_{k+1} - s_k)$ ,  $n_k^{*i} \in \mathcal{C}_i^*$  approximate the boundary integrals in (6.9). Consequently the vector  $S_m$  is strictly inside of the cone  $\mathcal{C}_i^*$  and in the limit converges to the centre

of mass of  $\gamma_i$  computed with respect of the origin (the vertex of the cone). In view of (6.10) there is a diameter of  $\mathbb{S}^2$  strictly inside of both dual cones  $\mathcal{C}_1^*$  and  $\mathcal{C}_2^*$ .

Without loss of generality we assume that the diameter passes through the north and south poles. Now we can apply Aleksandrov's moving plane method and Serrin's boundary point lemma to finish the proof. Let  $\Pi_t$  be the family of planes containing  $x_1$  axis and  $t$  measures the angle between  $\Pi_t$  and  $x_3$  axis.

Now start rotating  $\Pi_t$  about  $x_1$  axis starting from a position when  $\Pi_t$  is a support hyperplane to either of the cones  $\mathcal{C}_1^*, \mathcal{C}_2^*$  and  $\Pi_t \cap \mathcal{C}_i^* \neq \emptyset, i = 1, 2$ .

**Case 1:** If the first touch of  $\mathcal{M}$  and its reflection  $\tilde{\mathcal{M}}$  with respect to the plane  $\Pi_t$  occurs at some interior point of  $\mathcal{M}$ . Then from the maximum principle it follows that  $\mathcal{M} = \tilde{\mathcal{M}}$ .

By Lemma 6.4, both dual cones are strictly convex. Moreover, we claim that for  $\delta$  small the cones generated by  $\mathcal{M} \cap \partial B_{1-\delta}$  are convex, otherwise the inflection point would propagate to  $\mathcal{C}_i^*$ .

The two remaining possibilities are:

**Case 2:** if the first touch of  $\mathcal{M}$  and its reflection  $\tilde{\mathcal{M}}$  occurs at some boundary point where  $\partial\mathcal{M}$  is perpendicular to  $\Pi_t$ ,

**Case 3:** if the first touch of  $\mathcal{M}$  and its reflection  $\tilde{\mathcal{M}}$  occurs at some boundary point where  $\partial\mathcal{M}$  is not lying on  $\Pi_t$ .

We cannot directly apply Serrin's boundary point lemma [Ser71] because  $\partial\mathcal{M}$  is only  $C^{1,1}$  by virtue of Lemma 6.4. However, from the fibering of  $\mathcal{M}$  near  $\partial\mathcal{M}$  we conclude that  $\tilde{g} \leq g$  near the contact point, where  $\tilde{g}$  is the support function of  $\tilde{\mathcal{M}}$ . Thus  $\tilde{u} = r\tilde{g} \leq rg = u$ . Hence applying Serrin's boundary point lemma to the harmonic functions  $\tilde{u}$  and  $u$  we conclude that  $\mathcal{M} = \tilde{\mathcal{M}}$ .

Choosing  $\Pi_t$  to be an arbitrary family passing through a line perpendicular to the diameter it follows that  $\gamma_1, \gamma_2$  are circles and (6.10) forces them to lie on parallel planes. Applying Corollary 2 [Sch83] we infer that  $\mathcal{M}$  is a part of catenoid.  $\square$

## 7. APPENDIX

This section contains some well known results about the solutions of the singular perturbation problem  $(\mathcal{P}_\varepsilon)$ . We begin with the uniform Lipschitz estimates of Luis Caffarelli, see [Caf95] for the proof.

**Proposition 7.1.** *Let  $\{u_\varepsilon\}$  be a family of solution of  $(\mathcal{P}_\varepsilon)$  then there is a constant  $C$  depending only on  $N, \|\beta\|_\infty$  and independent of  $\varepsilon$  such that*

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{\frac{1}{2}})} \leq C. \quad (7.1)$$

As a consequence we get that one can extract converging sequences  $\{u_{\varepsilon_n}\}$  of solutions of  $(\mathcal{P}_\varepsilon)$  such that the limit functions are stationary points of the Alt-Caffarelli problem.

**Proposition 7.2.** *Let  $u_\varepsilon$  be a family of solutions to  $(\mathcal{P}_\varepsilon)$  in a domain  $\mathcal{D} \subset \mathbb{R}^N$ . Let us assume that  $\|u_\varepsilon\|_{L^\infty(\mathcal{D})} \leq A$  for some constant  $A > 0$  independent of  $\varepsilon$ . For every  $\varepsilon_n \rightarrow 0$  there exists a subsequence  $\varepsilon_{n'} \rightarrow 0$  and  $u \in C_{loc}^{0,1}(\mathcal{D})$ , such that*

- (i)  $u_{\varepsilon_{n'}} \rightarrow u$  uniformly on compact subsets of  $\mathcal{D}$ ,
- (ii)  $\nabla u_{\varepsilon_{n'}} \rightarrow \nabla u$  in  $L^2_{loc}(\mathcal{D})$ ,
- (iii)  $u$  is harmonic in  $\mathcal{D} \setminus \partial\{u > 0\}$ .

**Proof.** See Lemma 3.1 [CLW97]. □

Next, we recall the estimates for the slopes of some global solutions.

**Proposition 7.3.** *Let  $u$  be as in Proposition 7.2. Then the following statements hold true:*

- (i)  $u$  is Lipschitz,
  - (ii) if  $u_{\varepsilon_j} \rightarrow u = \alpha x_1^+$  locally uniformly, then  $0 \leq \alpha \leq \sqrt{2M}$ , see Proposition 5.2 [CLW97],
  - (iii) if  $u_{\varepsilon_j} \rightarrow u = \alpha x_1^+ - \gamma x_1^- + o(|x|)$  and  $\gamma > 0$  then  $\alpha^2 - \gamma^2 = \sqrt{2M}$ , see Proposition 5.1 [CLW97].
- In this lemma the essential assumption is that  $\gamma > 0$ .

**Remark 7.4.** Observe that if  $u(x) = \alpha x_1^+ + \bar{\alpha} x_1^-$  then we must necessarily have that  $\alpha = \bar{\alpha} \leq \sqrt{2M}$ , see Proposition 5.3 [CLW97]. In this case the interior of the zero set of  $u$  is empty. Thus one might have wedge-like solution.

Using Proposition 7.1 we can extract a sequence  $u_{\varepsilon_j}$  for some sequence  $\varepsilon_j$  such that  $u_{\varepsilon_j} \rightarrow u$  uniformly in  $B_{\frac{1}{2}}$ , see Proposition 7.2. Let  $u$  be a limit and  $0 < \rho_j \downarrow 0$  and  $u_j(x) = \frac{u(x_0 + \rho_j x)}{\rho_j}$ ,  $x_0 \in \partial\{u > 0\}$ . Thanks to Proposition 7.3(i) we can extract a subsequence, still labeled  $\rho_j$ , such that  $u_j$  converges to some function  $u_0$  defined in  $\mathbb{R}^N$ . The function  $u_0$  is called a blow-up limit of  $u$  at the free boundary point  $x_0$  and it depends on  $\{\rho_j\}$ .

The two propositions to follow establish an important property of the blow-up limits, namely that the first and second blow-ups of  $u$  can be obtained from  $(\mathcal{P}_\varepsilon)$  for a suitable choice of parameter  $\varepsilon$ . Observe that the scaled function  $\nabla(u_{\varepsilon_j})_{\lambda_n}$  verifies the equation

$$\Delta(u_{\varepsilon_j})_{\lambda_j} = \frac{\lambda_j}{\varepsilon_j} \beta \left( \frac{\lambda_j}{\varepsilon_j} (u_{\varepsilon_j})_{\lambda_j} \right). \quad (7.2)$$

Taking  $\delta_j = \frac{\varepsilon_j}{\lambda_j} \rightarrow 0$  we see that  $(u_{\varepsilon_j})_{\lambda_j}$  is solution to  $\Delta u_{\delta_j} = \beta_{\delta_j}(u_{\delta_j})$ .

**Proposition 7.5.** *Let  $u_{\varepsilon_j}$  be a family of solutions to  $(\mathcal{P}_\varepsilon)$  in a domain  $\mathcal{D} \subset \mathbb{R}^N$  such that  $u_{\varepsilon_j} \rightarrow u$  uniformly on  $\mathcal{D}$  and  $\varepsilon_j \rightarrow 0$ . Let  $x_0 \in \mathcal{D} \cap \partial\{u > 0\}$  and let  $x_n \in \partial\{u > 0\}$  be such that  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . Let  $\lambda_n \rightarrow 0$ ,  $u_{\lambda_n}(x) = (1/\lambda_n)u(x_n + \lambda_n x)$  and  $(u_{\varepsilon_j})_{\lambda_n} = (1/\lambda_n)u_{\varepsilon_j}(x_n + \lambda_n x)$ . Assume that  $u_{\lambda_n} \rightarrow U$  as  $n \rightarrow \infty$  uniformly on compact subsets of  $\mathbb{R}^N$ . Then there exists  $j(n) \rightarrow \infty$  such that for every  $j_n \geq j(n)$  there holds that  $\varepsilon_j/\lambda_n \rightarrow 0$  and*

- $(u_{\varepsilon_{j_n}})_{\lambda_n} \rightarrow U$  uniformly on compact subsets of  $\mathbb{R}^N$ ,
- $\nabla(u_{\varepsilon_{j_n}})_{\lambda_n} \rightarrow \nabla U$  in  $L_{loc}^2(\mathbb{R}^N)$ ,
- $\nabla u_{\lambda_n} \rightarrow \nabla U$  in  $L_{loc}^2(\mathbb{R}^N)$ .

**Proof.** See Lemma 3.2 [CLW97]. □

Finally, recall that the result of previous proposition extends to the second blow-up.

**Proposition 7.6.** *Let  $u_{\varepsilon_j}$  be a solution to  $(\mathcal{P}_\varepsilon)$  in a domain  $\mathcal{D}_j \subset \mathcal{D}_{j+1}$  and  $\cup_j \mathcal{D}_j = \mathbb{R}^N$  such that  $u_{\varepsilon_j} \rightarrow U$  uniformly on compact sets of  $\mathbb{R}^N$  and  $\varepsilon_j \rightarrow 0$ . Let us assume that for some choice of positive numbers  $d_n$  and points  $x_n \in \partial\{U > 0\}$ , the sequence*

$$U_{d_n}(x) = \frac{1}{d_n} U(x_n + d_n x)$$

*converges uniformly on compact sets of  $\mathbb{R}^N$  to a function  $U_0$ . Let*

$$(u_{\varepsilon_j})_{d_n} = \frac{1}{d_n} u_{\varepsilon_j}(x_n + d_n x).$$

*Then there exists  $j(n) \rightarrow \infty$  such that for every  $j_n \geq j(n)$ , there holds  $\varepsilon_{j_n}/d_n \rightarrow 0$  and*

- $(u_{\varepsilon_{j_n}})_{d_n} \rightarrow U_0$  uniformly on compact subsets of  $\mathbb{R}^N$ ,
- $\nabla(u_{\varepsilon_j})_{d_n} \rightarrow \nabla U_0$  in  $L^2_{loc}(\mathbb{R}^N)$ .

**Proof.** See Lemma 3.3 [CLW97]. □

Next lemma contains one of the crucial estimates needed for the proof of Proposition 5.15.

**Lemma 7.7.** *Let  $u \geq 0$  be as in Proposition 7.2. Then*

$$\limsup_{x \rightarrow x_0, u(x) > 0} |\nabla u(x)| \leq \sqrt{2M}.$$

**Proof.** To fix the ideas we let  $x_0 = 0$  and  $l = \limsup_{x \rightarrow 0, u(x) > 0} |\nabla u(x)|$ . Suppose  $l > 0$ , otherwise we are done. Choose a sequence  $z_k \rightarrow 0$  such that  $u(z_k) > 0$  and  $|\nabla u(z_k)| \rightarrow l$ . Setting  $\rho_k = |y_k - z_k|$ , where  $y_k \in \partial\{u > 0\}$  is the nearest point to  $z_k$  on the free boundary and proceeding as in the proof of [ACF84b] Lemma 3.4 we can conclude that the blow-up sequence  $u_k(x) = \rho_k^{-1} u(z_k + \rho_k x)$  has a limit  $u_0$  (at least for a subsequence, thanks to Proposition 7.1) such that  $u_0(x) = l x_1, x_1 > 0$  in a suitable coordinate system. Moreover, by Proposition 7.5 it follows that  $u_0$  is a limit of some  $u_{\lambda_j}$  solving  $\Delta u_{\lambda_j} = \beta_{\lambda_j}(u_{\lambda_j})$  in  $B_{r_j}, r_j \rightarrow \infty$ . If there is a point  $z \in \{x_1 = 0\}$  and  $r > 0$  such that  $u_0 > 0$  in  $B_r(z) \cap \{x_1 < 0\}$  then near  $z$  we must have  $u_0(x) = l(x - z)_1^+ + l(x - z)_1^- + o(x - z)$ , see Remark 7.4. Applying the unique continuation theorem to  $u_0(x) - u_0(-x_1, x_2, \dots, x_n)$  we see that  $u_0 = l(-x_1)^+, x_1 < 0$ . Thus recalling Remark 7.4 again we infer that  $l \leq \sqrt{2M}$ . □

Finally, we mention a useful identity for the solutions  $u_\varepsilon$ , see equation (5.2) [CLW97]: Let  $u_\varepsilon$  be a solution of  $(\mathcal{P}_\varepsilon)$  then for any  $\phi \in C_0^\infty(B_1)$  there holds

$$\int \left( \frac{|\nabla u_\varepsilon|^2}{2} + \mathcal{B}(u_\varepsilon/\varepsilon) \right) \partial_1 \phi = \int \sum_k \partial_k u_\varepsilon \partial_1 u_\varepsilon \partial_k \phi. \quad (7.3)$$

## REFERENCES

- [Ale39] A. Alexandroff, *Über die Oberflächenfunktion eines konvexen Körpers. (Bemerkung zur Arbeit "Zur Theorie der gemischten Volumina von konvexen Körpern")*, Rec. Math. N.S. [Mat. Sbornik] **6(48)** (1939), 167–174 (Russian, with German summary). MR0001597 ↑22, 23

- [AC81] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144. MR618549 ↑2, 4, 18, 19, 22
- [ACF84a] H. W. Alt, L. A. Caffarelli, and A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282** (1984), no. 2, 431–461. MR732100 ↑15
- [ACF84b] ———, *A free boundary problem for quasilinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), no. 1, 1–44. MR752578 ↑29
- [Caf87] L. A. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are  $C^{1,\alpha}$* , Rev. Mat. Iberoamericana **3** (1987), no. 2, 139–162. MR990856 ↑17
- [Caf89] ———, *A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz*, Comm. Pure Appl. Math. **42** (1989), no. 1, 55–78. MR973745 ↑17, 20
- [Caf95] ———, *Uniform Lipschitz regularity of a singular perturbation problem*, Differential Integral Equations **8** (1995), no. 7, 1585–1590. MR1347971 ↑2, 27
- [CJK04] L. A. Caffarelli, D. Jerison, and C. E. Kenig, *Global energy minimizers for free boundary problems and full regularity in three dimensions*, Noncompact problems at the intersection of geometry, analysis, and topology, Contemp. Math., vol. 350, Amer. Math. Soc., Providence, RI, 2004, pp. 83–97. MR2082392 ↑2, 5
- [CKS00] L. A. Caffarelli, L. Karp, and H. Shahgholian, *Regularity of a free boundary with application to the Pompeiu problem*, Ann. of Math. (2) **151** (2000), no. 1, 269–292. MR1745013 ↑15, 16
- [CLW97] L. A. Caffarelli, C. Lederman, and N. Wolanski, *Uniform estimates and limits for a two phase parabolic singular perturbation problem*, Indiana Univ. Math. J. **46** (1997), no. 2, 453–489. MR1481599 ↑3, 5, 11, 13, 28, 29
- [CS05] L. Caffarelli and S. Salsa, *A geometric approach to free boundary problems*, Graduate Studies in Mathematics, vol. 68, American Mathematical Society, Providence, RI, 2005. MR2145284 ↑15
- [Fan96] Y. Fang, *Lectures on minimal surfaces in  $\mathbf{R}^3$* , Proceedings of the Centre for Mathematics and its Applications, Australian National University, vol. 35, Australian National University, Centre for Mathematics and its Applications, Canberra, 1996. MR1401344 ↑26
- [HHP11] L. Hauswirth, F. Hélein, and F. Pacard, *On an overdetermined elliptic problem*, Pacific J. Math. **250** (2011), no. 2, 319–334. MR2794602 ↑4, 5
- [DJKP16] D. Jerison and K. Perera, *Higher critical points in a free boundary problem*, Preprint, arXiv:1610.00799 (2016). ↑5
- [JS15] D. Jerison and O. Savin, *Some remarks on stability of cones for the one-phase free boundary problem*, Geom. Funct. Anal. **25** (2015), no. 4, 1240–1257. MR3385632 ↑2
- [KLT13] D. Khavinson, E. Lundberg, and R. Teodorescu, *An overdetermined problem in potential theory*, Pacific J. Math. **265** (2013), no. 1, 85–111. MR3095114 ↑5
- [LR88] R. Langevin and H. Rosenberg, *A maximum principle at infinity for minimal surfaces and applications*, Duke Math. J. **57** (1988), no. 3, 819–828. MR975123 ↑22
- [Nit89] J. C. C. Nitsche, *Lectures on minimal surfaces. Vol. 1*, Cambridge University Press, Cambridge, 1989. Introduction, fundamentals, geometry and basic boundary value problems; Translated from the German by Jerry M. Feinberg; With a German foreword. MR1015936 ↑24
- [Nit85] ———, *Stationary partitioning of convex bodies*, Arch. Rational Mech. Anal. **89** (1985), no. 1, 1–19. MR784101 ↑4, 26
- [Nit62] ———, *A characterization of the catenoid*, J. Math. Mech. **11** (1962), 293–301. MR0137043 ↑26
- [Rez92] A. G. Reznikov, *Linearization and explicit solutions of the minimal surface equation*, Publ. Mat. **36** (1992), no. 1, 39–46. MR1179600 ↑23
- [RV95] A. Ros and E. Vergasta, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geom. Dedicata **56** (1995), no. 1, 19–33. MR1338315 ↑5

- [Sch14] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Second expanded edition, Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. MR3155183 ↑24
- [Sch83] R. M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), no. 4, 791–809 (1984). MR730928 ↑27
- [Ser71] J. Serrin, *A symmetry problem in potential theory*, Arch. Rational Mech. Anal. **43** (1971), 304–318. MR0333220 ↑27
- [Spr83] J. Spruck, *Uniqueness in a diffusion model of population biology*, Comm. Partial Differential Equations **8** (1983), no. 15, 1605–1620. MR729195 ↑3, 5, 8
- [Tra14] M. Traizet, *Classification of the solutions to an overdetermined elliptic problem in the plane*, Geom. Funct. Anal. **24** (2014), no. 2, 690–720. MR3192039 ↑5
- [Wei03] G. S. Weiss, *A singular limit arising in combustion theory: fine properties of the free boundary*, Calc. Var. Partial Differential Equations **17** (2003), no. 3, 311–340. MR1989835 ↑2, 9, 14

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, PETER TAIT GUTHRIE ROAD, EH9 3FD, EDINBURGH, UK  
*E-mail address:* `aram6k@gmail.com`